A UNITED STATES DEPARTMENT OF COMMERCE PUBLICATION

SECTION B

JOURNAL OF RESEARCH

of the NATIONAL BUREAU OF STANDARDS

JNBBAU 78B (1) 1-62, Jan.-Mar. 1974

Mathematical Sciences

U.S. DEPARTMENT OF COMMERCE
National Bureau of Standards

Journal of Research

of the

National Bureau of Standards

B. Mathematical Sciences

Jan.–Mar. • 1974

VOLUME • 78B

NUMBER • 1

Editor: Morris Newman

Associate Editor: F. W. Olver

Publication dates: Mar. 10, June 10, Sept. 10, Dec. 10, 1974

J. Res. Nat. Bur. Stand. (U.S.), B

JOURNAL OF RESEARCH

The National Bureau of Standards Journal of Research reports research and development in the fields of activity shown at right. Also included from time to time are survey articles on topics closely related to the Bureau's scientific and technical program.

The Journal is published in two separate sections as follows:

A. Physics and Chemistry

Contains papers of interest primarily to scientists working in these fields. Issued six times a year. Annual subscription: domestic, $17.00; foreign, $21.25. Single copy, varies.

B. Mathematical Sciences

Presents studies and compilations designed mainly for the mathematician and the theoretical physicist. Issued quarterly. Annual subscription: domestic, $9.00; foreign, $11.25. Single copy $2.25.

The National Bureau of Standards[1] was established by an act of Congress March 3, 1901. The Bureau's overall goal is to strengthen and advance the Nation's science and technology and facilitate their effective application for public benefit. To this end, the Bureau conducts research and provides: (1) a basis for the Nation's physical measurement system, (2) scientific and technological services for industry and government, (3) a technical basis for equity in trade, and (4) technical services to promote public safety. The Bureau consists of the Institute for Basic Standards, the Institute for Materials Research, the Institute for Applied Technology, the Institute for Computer Sciences and Technology, and the Office for Information Programs.

THE INSTITUTE FOR BASIC STANDARDS provides the central basis within the United States of a complete and consistent system of physical measurement; coordinates that system with measurement systems of other nations; and furnishes essential services leading to accurate and uniform physical measurements throughout the Nation's scientific community, industry, and commerce. The Institute consists of a Center for Radiation Research, an Office of Measurement Services and the following divisions:

Applied Mathematics — Electricity — Mechanics — Heat — Optical Physics — Nuclear Sciences[2] — Applied Radiation[2] — Quantum Electronics[3] — Electromagnetics[3] — Time and Frequency[3] — Laboratory Astrophysics[3] — Cryogenics[3].

THE INSTITUTE FOR MATERIALS RESEARCH conducts materials research leading to improved methods of measurement, standards, and data on the properties of well-characterized materials needed by industry, commerce, educational institutions, and Government; provides advisory and research services to other Government agencies; and develops, produces, and distributes standard reference materials. The Institute consists of the Office of Standard Reference Materials and the following divisions:

Analytical Chemistry — Polymers — Metallurgy — Inorganic Materials — Reactor Radiation — Physical Chemistry.

THE INSTITUTE FOR APPLIED TECHNOLOGY provides technical services to promote the use of available technology and to facilitate technological innovation in industry and Government; cooperates with public and private organizations leading to the development of technological standards (including mandatory safety standards), codes and methods of test; and provides technical advice and services to Government agencies upon request. The Institute consists of a Center for Building Technology and the following divisions and offices:

Engineering and Product Standards — Weights and Measures — Invention and Innovation — Product Evaluation Technology — Electronic Technology — Technical Analysis — Measurement Engineering — Structures, Materials, and Life Safety[4] — Building Environment[4] — Technical Evaluation and Application[4] — Fire Technology.

THE INSTITUTE FOR COMPUTER SCIENCES AND TECHNOLOGY conducts research and provides technical services designed to aid Government agencies in improving cost effectiveness in the conduct of their programs through the selection, acquisition, and effective utilization of automatic data processing equipment; and serves as the principal focus within the executive branch for the development of Federal standards for automatic data processing equipment, techniques, and computer languages. The Center consists of the following offices and divisions:

Information Processing Standards — Computer Information — Computer Services — Systems Development — Information Processing Technology.

THE OFFICE FOR INFORMATION PROGRAMS promotes optimum dissemination and accessibility of scientific information generated within NBS and other agencies of the Federal Government; promotes the development of the National Standard Reference Data System and a system of information analysis centers dealing with the broader aspects of the National Measurement System; provides appropriate services to ensure that the NBS staff has optimum accessibility to the scientific information of the world. The Office consists of the following organizational units:

Office of Standard Reference Data — Office of Technical Information and Publications — Library — Office of International Relations.

[1] Headquarters and Laboratories at Gaithersburg, Maryland, unless otherwise noted; mailing address Washington, D.C. 20234.
[2] Part of the Center for Radiation Research.
[3] Located at Boulder, Colorado 80302.
[4] Part of the Center for Building Technology.

U.S. DEPARTMENT OF COMMERCE
Frederick B. Dent, Secretary

NATIONAL BUREAU OF STANDARDS
Richard W. Roberts, Director

Order all publications from the Superintendent of Documents
U.S. Government Printing Office, Washington, D.C. 20402.

The Secretary of Commerce has determined that the publication of this periodical is necessary in the transaction of the public business required by law of this Department. Use of funds for printing this periodical has been approved by the Director of the Office of Management and Budget through June 30, 1976.

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 78B, No. 1, January–March 1974

A Class of Positive Stable Matrices*

David Carlson**

(September 11, 1973)

A square complex matrix is positive sign-symmetric if all its principal minors are positive, and all products of symmetrically-placed minors are nonnegative. It is proved that every positive sign-symmetric matrix is positive stable.

Key words: Positive stable matrix; sign-symmetry; spectrum.

1. NOTATION. For fixed n, let

$$Q_n = \{(i_1, i_2, \ldots, i_k) \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}.$$

If $\alpha = (i_1, i_2, \ldots, i_k) \in Q_n$, then $|\alpha| = k$. Given $A \in C^{n,n}$ and $\alpha, \beta \in Q_n$, by $A(\alpha, \beta)$ we mean the minor of A whose rows are indexed by α and whose columns are indexed by β. We can now formally define $A \in C^{n,n}$ to be *positive sign-symmetric* if

(1) $A(\alpha, \alpha) > 0$ for all $\alpha \in Q_n$,
(2) $A(\alpha, \beta) A(\beta, \alpha) \geq 0$ for all $\alpha, \beta \in Q_n$, $|\alpha| = |\beta|$.

It is obvious that hermitian positive definite matrices and totally positive matrices are positive sign-symmetric. Also, it is well known that they have all positive characteristic roots (see also [9][1]). This last is not true for all positive sign-symmetric matrices; as an example, take

$$A = \begin{pmatrix} 4 & 1 & 1 \\ 2 & 4 & 2 \\ 2 & 1 & 4 \end{pmatrix}.$$

The characteristic roots of this matrix are, approximately, 6.85 and $2.58 \pm 0.28i$.

It is perhaps of interest, however, that all positive sign-symmetric matrices are *positive stable*, i.e., all their characteristic roots have positive real parts.

THEOREM. *Every complex matrix which is positive sign-symmetric is positive stable.*

PROOF. Given $B \in C^{n,n}$ which is positive sign-symmetric. Let $S = B^2$; by the Cauchy-Binet formula and (1) and (2),

$$S(\alpha, \alpha) = \Sigma_{|\beta| = |\alpha|} B(\alpha, \beta) B(\beta, \alpha) > 0 \text{ for all } \alpha \in Q_n.$$

By a result of Fiedler and Pták [5], since all principal minors of S are positive, no real characteristic root of S is nonpositive. Since the characteristic roots of S are the squares of the characteristic roots of B, B can have no characteristic roots on the imaginary axis.

AMS Subject Classification: 15A18, 15A57, 65F15.
*An invited paper.
**Present address: Mathematics Department Oregon State University, Corvallis, Oregon 97331.

[1] Figures in brackets indicate the literature references at the end of this paper.

Now let $A \in C^{n,n}$ be a positive sign-symmetric matrix. Since all its principal minors are positive, by a result of Fisher and Fuller [6] and Ballantine [1], there exists a $D = \text{diag}\,(d_1, d_2, \ldots, d_n) \in C^{n,n}$, $d_i > 0$, $i = 1, \ldots, n$, for which AD is positive stable. Let $D_t = (1-t)\,I + tD$, $0 \leq t \leq 1$; all D_t are diagonal matrices, with positive diagonal entries, and $D_0 = I$, $D_1 = D$. It follows that each matrix AD_t is positive sign-symmetric, and hence can have no characteristic roots on the imaginary axis. Since AD is positive stable, and the characteristic roots of AD_t are continuous functions of t, A must also be positive stable.

We define $A \in C^{n,n}$ to be *positive weakly sign-symmetric* if (1) and

(3) $A(\alpha, \beta)\,A\,(\beta, \alpha) \geq 0$ for all $\alpha, \beta \in Q_n$, $|\alpha| = |\beta| = |\alpha \cap \beta| + 1$,

i.e., whenever all but one of the indices in α and β are common to both. A number of interesting inequalities are known to hold for the principal minors of such matrices; cf. [3], [4], [8]. It is known that nonsingular M-matrices, as well as hermitian positive definite and totally positive matrices, are positive weakly sign-symmetric (cf. [2]); and all these matrices are positive stable. We conjecture that all positive weakly sign-symmetric matrices are also positive stable. The Routh-Hurwitz conditions (cf. [7], p. 194–5) can be used to prove that the conjecture is true for $n \leq 4$ (the case $n = 3$ is also proved in [2]). An analogous proof for the general case appears hopelessly complicated.

References

[1] Ballantine, C. S., Stabilization by a diagonal matrix, Proc. Amer. Math. Soc. **25**, 728–734 (1970).

[2] Carlson, David, Weakly sign-symmetric matrices and some determinantal inequalities, Colloq. Math. **17**, 123–129 (1967).

[3] Fan, Ky, Subadditive functions on a distributive lattice and an extension of Szász's inequality, J. Math. Anal. Appl. **18**, 262–268 (1967).

[4] Fan, Ky, An inequality for subadditive functions on a distributive lattice, with application to determinantal inequalities, Lin. Alg. Appl. **1**, 33–38 (1968).

[5] Fiedler, M., and Pták, V., On matrices with non-positive off-diagonal elements and positive principal minors, Czech. Math. Journal **12**, 382–400 (1962).

[6] Fisher, M. E., and Fuller, A. T., On the stabilization of matrices and the convergence of linear iterative processes, Proc. Camb. Philos. Soc. **54**, 417–425 (1958).

[7] Gantmacher, F. R., The Theory of Matrices, Chelsea, New York, 1959, Vol. II, 276 p.

[8] Koteljanskiĭ, D. M., A property of sign-symmetric matrices, Uspehi Mat. Nauk (N.S.) **8**, 163–7 (1953).

[9] Koteljanskiĭ, D. M., Some sufficient conditions for reality and simplicity of the spectrum of a matrix, Mat. Sb. N.S. **36**, 163–168 (1955).

(Paper 78B1–391)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 78B, No. 1, January–March 1974

The Smith Normal Form of a Partitioned Matrix

Morris Newman

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(September 11, 1973)

It is shown that if

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ 0 & M_{22} & \cdots & M_{2t} \\ & \cdots & & \\ 0 & 0 & \cdots & M_{tt} \end{bmatrix}$$

is a matrix over a principal ideal ring R such that the matrices M_{ii} are square and have pairwise relatively prime determinants, then the Smith normal form of M is the same as the Smith normal form of

$M_{11} \dot{+} M_{22} \dot{+} \cdots \dot{+} M_{tt}$.

Key words: Elementary divisors; invariant factors; partitioned matrices; Smith normal form.

Let R be a principal ideal ring. Let A be an $r \times r$ matrix over R, B an $s \times s$ matrix over R. It is well known that the elementary divisors of $A \dot{+} B$ are the elementary divisors of A together with the elementary divisors of B, which allows us to reconstruct the Smith Normal Form (hereafter abbreviated S.N.F.) of $A \dot{+} B$ from the invariant factors of A and of B (see [1],[1] for example). There is a noteworthy instance which merits special attention: namely, when the determinants of A and B are relatively prime. This note is devoted to this case.

We let $S(M)$ denote the S.N.F. of any matrix M over R, and I_n denote the identity matrix of order n. I will denote an identity matrix of unspecified order.

We first prove

THEOREM 1: *Suppose that* $(det(\mathrm{A}), det(\mathrm{B})) = 1$, *and that*

$$\mathrm{S(A)} = diag(\alpha_1, \alpha_2, \ldots, \alpha_r),$$

$$\mathrm{S(B)} = diag\,(\beta_1, \beta_2, \ldots, \beta_s),$$

so that $\alpha_1, \alpha_2, \ldots, \alpha_r$ *are the invariant factors of* A, $\beta_1, \beta_2, \ldots, \beta_s$ *the invariant factors of* B; *and assume for definiteness that* $r \leq s$. *Then*

$$(1)\quad \mathrm{S(A \dot{+} B)} = \mathrm{I}_r \dot{+} diag\,(\beta_1, \beta_2, \ldots, \beta_{s-r}) \dot{+} diag\,(\alpha_1\beta_{s-r+1}, \alpha_2\beta_{s-r+2}, \ldots, \alpha_s\beta_r).$$

AMS Subject Classification: 1530, 1545, 1548.

[1] Figures in brackets indicate the literature references at the end of this paper.

PROOF: A moment's consideration shows that the expression given in (1) for $S(A \dot{+} B)$ is just $S(I_s \dot{+} A)\, S(I_r \dot{+} B)$. Now

$$A \dot{+} B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I_s \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & B \end{bmatrix}$$

$$= (A + I_s)\,(I_r \dot{+} B).$$

But $A \dot{+} I_s$ and $I_r \dot{+} B$ have relatively prime determinants, and it is known that if M, N are matrices over R of the same size such that $(\det(M), \det(N)) = 1$, then $S(MN) = S(M)S(N)$ (see [2]). It follows that

$$S(A \dot{+} B) = S(A \dot{+} I_s)\, S\,(I_r \dot{+} B)$$

$$= S(I_s + A)\, S\,(I_r \dot{+} B).$$

This concludes the proof.

Now let T be any $r \times s$ matrix over R. Then provided that $(\det(A), \det(B)) = 1$, the next result shows that T plays no part in determining the S.N.F. of

$$\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}.$$

Specifically, we prove

THEOREM 2: *Suppose that* $(det(\mathrm{A}), det(\mathrm{B})) = 1$. *Then the* S.N.F. *of*

$$\begin{bmatrix} \mathrm{A} & \mathrm{T} \\ 0 & \mathrm{B} \end{bmatrix}$$

is the same as the S.N.F. *of*

$$\begin{bmatrix} \mathrm{A} & 0 \\ 0 & \mathrm{B} \end{bmatrix}.$$

PROOF: Let A^{adj} be the adjoint of A, B^{adj} the adjoint of B, so that A^{adj}, B^{adj} are matrices over R satisfying

$$AA^{\mathrm{adj}} = A^{\mathrm{adj}}A = \det(A) \cdot I_r,$$

$$BB^{\mathrm{adj}} = B^{\mathrm{adj}}B = \det(B) \cdot I_s.$$

Since $(\det(A), \det(B)) = 1$, elements α, β of R exist such that

$$\alpha \det(A) + \beta \det(B) = 1.$$

Now consider the equation

$$\begin{bmatrix} I_r & X \\ 0 & I_s \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I_r & Y \\ 0 & I_s \end{bmatrix} = \begin{bmatrix} A & T \\ 0 & B \end{bmatrix}. \tag{2}$$

Then (2) holds if and only if

$$T = AY + XB. \tag{3}$$

But (3) may be satisfied by choosing

$$X = \beta T B^{(\mathrm{adj})}, \quad Y = \alpha A^{(\mathrm{adj})} T.$$

Thus (2) has a solution in matrices X, Y over R, and it follows that $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are equivalent, and hence have the same S.N.F. This concludes the proof.

We remark that because of Theorem 1, the S.N.F. of $\begin{bmatrix} A & T \\ 0 & B \end{bmatrix}$ is completely determined by the invariant factors of A and the invariant factors of B, when $(\det(A), \det(B)) = 1$.

This result is definitely false if the determinant condition is removed. For example, the S.N.F. of $\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$ is $\begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix}$, but the S.N.F. of $\begin{bmatrix} 4 & 1 \\ 0 & 6 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 24 \end{bmatrix}$.

We note in passing that if

$$S(A) = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_r),$$

then

$$S(A \dot{+} A \dot{+} \ldots \dot{+} A) = \alpha_1 I_k \dot{+} \alpha_2 I_k \dot{+} \ldots \dot{+} \alpha_r I_k,$$

where there are k replicas of A in the direct sum.

Theorem 2 may be generalized as follows:

THEOREM 3: *Let* M *be a matrix over* R, *and suppose that* M *may be partitioned as*

$$\mathrm{M} = \begin{bmatrix} \mathrm{M}_{11} & \mathrm{M}_{12} & \ldots & \mathrm{M}_{1t} \\ 0 & \mathrm{M}_{22} & \ldots & \mathrm{M}_{2t} \\ & \ldots & & \\ 0 & 0 & \ldots & \mathrm{M}_{tt} \end{bmatrix},$$

where the matrices M_{ii} *are square and have pairwise relatively prime determinants. Then the* S.N.F. *of* M *is determined by the invariant factors of the* M_{ii};

$$\mathrm{S}(\mathrm{M}) = \mathrm{S}(\mathrm{M}_{11} \dot{+} \mathrm{M}_{22} \dot{+} \ldots \dot{+} \mathrm{M}_{tt}).$$

PROOF: Put

$$A = M_{11},$$

$$T = [M_{12}, \ldots, M_{1t}],$$

$$B = \begin{bmatrix} M_{22} & \ldots & M_{2t} \\ & \ldots & \\ 0 & \ldots & M_{tt} \end{bmatrix}.$$

Then

$$M = \begin{bmatrix} A & T \\ 0 & B \end{bmatrix},$$

and $(\det(A), \det(B)) = 1$. By Theorem 2 and the result on the multiplicativity of the S.N.F.,

$$S(M) = S\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)$$

$$= S\left(\begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}\right) S\left(\begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}\right),$$

since $(\det(A), \det(B)) = 1$. Repeating this procedure with the matrix B, we ultimately obtain

$$S(M) = S(M_{11} \dot{+} I) S(I \dot{+} M_{22} \dot{+} I) \ldots S(I \dot{+} M_{tt})$$

$$= S(M_{11} \dot{+} M_{22} \dot{+} \ldots \dot{+} M_{tt}).$$

This concludes the proof.

References

[1] Newman, Morris, Integral Matrices, Academic Press, New York (1972).
[2] ———, On the Smith normal form, J. Res. Nat. Bur. Stand. (U.S.), 75B (Math. Sci.), Nos. 1&2, 81–84 (Jan.–June 1971).

(Paper 78B1–392)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 78B, No. 1, January–March 1974

A Lyapunov Theorem for Angular Cones

Charles R. Johnson*

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(August 9, 1973)

The well known theorem of Lyapunov is generalized to characterize matrices whose spectra lie in a given open convex angular sector. Related facts about positive definite matrices, the polar decomposition and matrices with cramped spectra are also given.

Key words: Cramped; field of values; open positive convex cone; polar decomposition; positive definite; spectrum; square matrix.

A theorem of Lyapunov which has several equivalent formulations characterizes the matrices whose eigenvalues lie in the right half-plane. By considering one of the formulations from a new point of view, a characterization of the matrices whose eigenvalues lie in an arbitrary open positive convex cone is obtained. The positive cone of the characterization generalizes the right half-plane of Lyapunov's Theorem.

Throughout we shall consider square matrices $A, B, C \ldots$ over the complex field. Denote the field of values of A by $F(A)=\{x^*Ax \mid \|x\|=1\}$ and the set of eigenvalues (spectrum) of A by $\sigma(A)$. It is well known that $F(A)$ is a compact convex subset of the complex plane which contains $\sigma(A)$ and that $F(A)$ coincides with the convex hull of $\sigma(A)$ when A is normal. We shall also employ the concept of the angular field $F_{\text{ang}}(A)=\{x^*Ax \mid x \neq 0\}$ mentioned by Wielandt [5][1] and the angular spectrum $\sigma_{\text{ang}}(A)=\left\{\sum_{i=1}^{n} \alpha_i\lambda_i \mid \sigma(A)=\{\lambda_1, \ldots, \lambda_n\};\ \alpha_i \geq 0,\ i=1, \ldots, n;\ \text{and}\ \sum_{i=1}^{n} \alpha_i > 0\right\}$. $F_{\text{ang}}(A)$ and $\sigma_{\text{ang}}(A)$, respectively, are just the smallest positive convex cones which are anchored at the origin and contain $F(A)$ and $\sigma(A)$. Finally let R signify the open right complex half-plane and Σ be the class of positive definite hermitian matrices.

The three versions of Lyapunov's Theorem of interest are

(1) [Strong Form [3]]
$\sigma(A) \subset R$ if and only if for each $H \in \Sigma$ there is a $G \in \Sigma$ such that $GA+A^*G=H$;

(2) [Weak Form]
$\sigma(A) \subset R$ if and only if there is a $G \in \Sigma$ such that $GA+A^*G \in \Sigma$; and

(3) [Field of Values Formulation]
$\sigma(A) \subset R$ if and only if there is a $G \in \Sigma$ such that $F(GA) \subset R$.

That (2) and (3) are equivalent follows immediately from the fact that the field of values of $\frac{1}{2}(GA+A^*G)$ is just the projection of $F(GA)$ onto the real axis. For a discussion of the equivalence of (1) and (2) see [3, 4].

AMS Subject Classification: 15–A18, –A42, –A57, –A63.

*This work was done while the author was a National Academy of Sciences–National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234.

[1] Figures in brackets indicate the literature references at the end of this paper.

Formulation (3) is of interest in that it points out that Lyapunov's Theorem relates a region of containment for $\sigma(A)$ to a region of containment for the field of values of a positive definite multiple of A. In particular the two conic regions of containment are the same. We shall show that the R of Lyapunov's theorem (3) can be replaced by an arbitrary open positive convex cone Γ (Γ shall be such throughout). This then characterizes $\{A \mid \sigma(A) \subset \Gamma\}$ and relates the eigenvalues and field of values for arbitrary matrices. It should be noted that the following is of interest only when Γ describes an arc of less than or equal to π.

THEOREM A: *The set* $\sigma(A) \subset \Gamma$ *if and only if there is a* $G \epsilon \Sigma$ *such that* $F(GA) \subset \Gamma$.

We present the proof of the theorem using 3 lemmas.

Let Co denote the convex closure and d the distance function.

LEMMA 1: *Given an arbitrary square complex matrix* A *and a positive number* ϵ, *there exists an invertible matrix* S *such that*

$$d\Big(F(SAS^{-1}), Co(\sigma(A))\Big) < \epsilon.$$

(The field of values can be made close to the convex hull of the spectrum via similarity.)

PROOF: Reduce A to Jordan form J by similarity. If J is diagonal, we are done. If not, the superdiagonal 1's may be made arbitrarily small by diagonal similarities. Thus $F(D^{-1}JD)$ may be made arbitrarily close to the convex hull of the diagonal of J which is just $Co(\sigma(A))$.

LEMMA 2: *Let det* $B \neq 0$. *Then*

$$F_{ang}(B^*AB) = F_{ang}(A).$$

(The invariance of the angular field under congruence will mean that $F(B^*AB) \subset \Gamma$ if and only if $F(A) \subset \Gamma$.)

PROOF: $x^*B^*ABx = y^*Ay$ and $y = Bx \neq 0$ if and only if $x \neq 0$.

LEMMA 3: *If* $G \epsilon \Sigma$, *then*

$$\sigma(A) \subset \sigma_{ang}(A) \subset F_{ang}(GA).$$

PROOF: It suffices to show $\sigma(A) \subset F_{ang}(GA)$. Since $G \epsilon \Sigma$, G may be written $G = P^*P$, $\det P \neq 0$. Then,

$$\sigma(A) = \sigma(PAP^{-1}) = \sigma((P^{-1})^*GAP^{-1}) \subset F((P^{-1})^*GAP^{-1}) \subset F_{ang}((P^{-1})^*GAP^{-1}) = F_{ang}(GA).$$

PROOF (of theorem): Suppose $\sigma(A) \subset \Gamma$; choose S by lemma 1 so that

$$d(F(SAS^{-1}), Co(\sigma(A))) < \min_{c \epsilon \overline{\Gamma}} (Co(\sigma(A)), c),$$

which is positive. This means $F(SAS^{-1}) \subset \Gamma$. By lemma 2, $F(S^*(SAS^{-1})S) \subset \Gamma$ also, so that we have $F(GA) \subset \Gamma$ where $G = S^*S \epsilon \Sigma$.

For the converse suppose $F(GA) \subset \Gamma$, $G \epsilon \Sigma$. Then $F_{ang}(GA) \subset \Gamma$ and by lemma 3, $\sigma(A) \subset F_{ang}(GA) \subset \Gamma$ and the theorem is proven.

That Σ is the only class of matrices enjoying the property of lemma 3 is of individual interest.

THEOREM B: $\sigma(A) \subset F_{ang}(GA)$ *for all matrices* A *if and only if* $G \epsilon \Sigma$.

PROOF: It suffices to show that $\sigma(A) \subset F_{ang}(GA)$ for all A implies $G \epsilon \Sigma$ since the converse has been proven in lemma 3.

First let A run through Σ and notice that $\sigma(AG^*) \subset F_{ang}(GAG^*) = F_{ang}(A) =$ the positive real axis. This means that $\sigma(GA)$ is real and positive for all $A \epsilon \Sigma$. Now write G in its polar form $G=UH$ where U is unitary and $H \epsilon \Sigma$. We may take $A=H^{-1} \epsilon \Sigma$ and then $GA=UHH^{-1}=U$. Since $\sigma(U)=\sigma(GA)$ is real and positive, U can only be the identity. Thus $G=H \epsilon \Sigma$.

Several remarks now follow from the preceding.

COROLLARY 1: $\sigma_{ang}(A) = \bigcap_{G \epsilon \Sigma} F_{ang}(GA)$.

PROOF: By theorem B $\sigma_{ang}(A) \subset \bigcap_{G \epsilon \Sigma} F_{ang}(GA)$; but in theorem A we may choose Γ as close as we like to $\sigma_{ang}(A)$ which proves the equality.

COROLLARY 2: *If* $A^*C \epsilon \Sigma$ (*or equivalently* $CA^{-1} \epsilon \Sigma$), *then* $\sigma(A) \subset F_{ang}(C)$.

PROOF: Suppose $CA^{-1}=H \epsilon \Sigma$, then

$$\sigma(A) \subset F_{ang}(HA) = F_{ang}(C).$$

COROLLARY 3: *Suppose* $A = HU$, *where* U *is unitary and* $H \epsilon \Sigma$ *is the polar decomposition of* A, *then*

$$\sigma_{ang}(A) \subset F_{ang}(U) \subset F_{ang}(A).$$

PROOF: By assumption $AU^*=H \epsilon \Sigma$ or, equivalently, $UA^{-1} \epsilon \Sigma$ which implies $\sigma_{ang}(A) \subset F_{ang}(U)$ by corollary 2. Since U is normal, $F_{ang}(U) = \sigma_{ang}(U)$ which is contained in $F_{ang}(HU) = F_{ang}(A)$ by Theorem B.

In order to characterize the matrices whose eigenvalues lie in a half-plane, we define a matrix A to be cramped if (1) $\det A \neq 0$ and (2) the set $\left\{ \frac{\lambda}{|\lambda|} \;\middle|\; \lambda \epsilon \sigma(A) \right\}$ lies on an arc of the unit circle of less than 180°. It is clear that this is equivalent to saying the maximum pairwise difference of the arguments of the eigenvalues is less than π (and $0 \notin \sigma(A)$) or there is a θ such that $\sigma(e^{i\theta}A) \subset R$. We then have

COROLLARY 4: *If* $\det A \neq 0$, *the following statements are equivalent:*

(i) A *is cramped*;

(ii) $0 \notin \sigma_{ang}(A)$;

(iii) *there is a* $G \epsilon \Sigma$ *such that* $0 \notin F(GA)$;

(iv) *there is a* $G \epsilon \Sigma$ *such that* $F_{ang}(GA)$ *is not the entire complex plane and* A *does not have two characteristic roots which determine a line segment containing* 0.

PROOF: The equivalence of (i) and (ii) is immediate and the equivalence of (i) and (iii) follows from theorem A. If $F_{ang}(GA)$ is not the entire complex plane, it can be at most a closed half-plane which means $\sigma_{ang}(A)$ is at most a closed half-plane by theorem B. It then follows that (iv) implies (i). Since $0 \notin F(GA)$ implies $0 \notin F_{ang}(GA)$ and that A is cramped, we have that (iii) implies (iv) and the corollary is complete.

COROLLARY 5: *If* $G \epsilon \Sigma$ *and* GA *is hermitian* (*resp. hermitian positive definite*), *then the roots of* A *are real* (*resp. real and positive*).

PROOF: An application of Corollary 2. This last corollary is known by other means: see [2, 5].

In order to facilitate a final corollary which translates theorem A into a representation theorem, let $F(\Gamma) = \{A \mid F(A) \subset \Gamma\}$ and $S(\Gamma) = \{A \mid \sigma(A) \subset \Gamma\}$ where Γ is as before. It is clear that $F(\Gamma) \subset S(\Gamma)$.

COROLLARY 6. $S(\Gamma) = \Sigma \cdot F(\Gamma)$

PROOF: Suppose $A \epsilon \Sigma \cdot F(\Gamma)$, then $A=PB$ where $P \epsilon \Sigma$ and $F(B) \subset \Gamma$. Then $P^{-1} \epsilon \Sigma$ and $P^{-1}A = B \epsilon F(\Gamma)$ which implies $\sigma(A) \subset \Gamma$ by theorem A and means $A \epsilon S(\Gamma)$.

If $A \epsilon S(\Gamma)$, then also by theorem A, we have $F(GA) \subset \Gamma$ for some $G \epsilon \Sigma$. Since $G^{-1} \epsilon \Sigma$ and $A = G^{-1}(GA)$ we have $A \epsilon \Sigma \cdot F(\Gamma)$ to complete the proof.

Finally one question is suggested and left for further study by theorem A. How may we characterize the B's such that $F(B) \subset \Gamma$, $B=GA$ where $G \epsilon \Sigma$ and $\sigma(A) \subset \Gamma$, or alternatively how do

we characterize the G's in Σ such that $F(GA) \subset \Gamma$. When $\Gamma = R$, this question is answered in part by Lyapunov's theorem (1) and the invertibility of the Lyapunov operator. In this case any positive definitite real part may be obtained, but with each real part only one imaginary part may be obtained. Thus many B with $F(B) \subset R$ are omitted from the range.

References

[1] Givens, W., Fields of values of a matrix, Proc. AMS **3**, 206–209 (1952).
[2] Taussky-Todd, O., Problem 4846, Am. Math. Monthly **66**, 427 (1959).
[3] Taussky-Todd, O., Matrices C with $C^n \to 0$, J. of Algebra **1**, 5–10 (1964).
[4] Taussky-Todd, O., On stable matrices, Colloques Internationaux Du Centre National De La Recherche Scientifique No. 165; Programmation En Mathematiques Numeriques, Sept. 1966.
[5] Wielandt, H., On the eigenvalues of A+B and AB, J. Res. Nat. Bur. Stand. (U.S.), 77B (Math. Sci.) Nos. 1 & 2, 61–63 (Jan.–June 1973).
[6] Wigner, E. P., On weakly positive matrices, Can. J. of Math. **15**, 313–317 (1963).
[7] Williams, J. P., Similarity and the numerical range, J. Math. Anal. and Applics. **26**, 307–314 (1969).

(Paper 78B1–393)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 78B, No. 1, January–March 1974

Second, Third, and Fourth Order *D*-Stability*

Charles R. Johnson

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(August 30, 1973)

For $n=2$, 3, and 4, conditions are given for the real n by n D-stable matrices. The 3 by 3 sufficient condition is easily checkable and reveals to be D-stable a class of matrices which is not included in any known, general sufficient condition.

Key words: D-stable; positive stable matrix; spectrum.

The concept of D-stability was originally introduced in the economic literature by Arrow and McManus [1][1] with a stronger definition. We shall adopt the definition of fairly common current usage. Let $M_n(R)$ denote the class of n by n matrices over the real field and denote by $\sigma(A)$ the *spectrum* of $A \epsilon M_n(R)$. The matrix $A \epsilon M_n(R)$ is called (*positive*) *stable* if $\lambda \epsilon \sigma(A)$ implies $Re(\lambda) > 0$. We shall denote the multiplicative group of diagonal matrices with positive diagonal entries in $M_n(R)$ by D_n.

DEFINITION: $A \epsilon M_n(R)$ *is called* D-*stable if* DA *is stable for all* $D \epsilon D_n$.

Several sufficient and some necessary conditions for D-stability are known; however, no general characterization is yet known. In this note we present conditions on the D-stable matrices in $M_n(R)$ when $n=2$, 3, and 4. Only one of the known necessary conditions will be of interest to us here.

DEFINITION: $A \epsilon M_n(R)$ *belongs to the class* P_0 [2] *if and only if for each* $k=1, \ldots, n$ *all* k *by* k *principal minors of* A *are nonnegative. If also, at least one principal minor of each order is positive, then* $A \epsilon P_0^+$.

The best necessary condition for D-stability seems to be

THEOREM 0: [4, 5] *If* $A \epsilon M_n(R)$ *is* D-*stable, then* $A \epsilon P_0^+$.

The converse of theorem 0 is, in general, far from valid. However, for $n=2$ we have

THEOREM 1: $A \epsilon M_2(R)$ *is* D-*stable if and only if* $A \epsilon P_0^+$.

PROOF: The necessity follows from theorem 0. Suppose $A \epsilon P_0^+ \cap M_2(R)$ and that D is an arbitrary element of D_2. Then DA has positive trace and positive determinant. Since $DA \epsilon M_2(R)$, this means that DA is positive stable and that A is D-stable which completes the proof.

For our remaining work we shall employ the stability theorem of Routh and Hurwitz [3]. For $A \epsilon M_n(R)$ denote the sum of the $\binom{n}{k}$ principal minors of order k by $E_k(A)$. Define the Routh-Hurwitz matrix Ω by

---

AMS Subject Classification: 15–A18, 15–A57, 65–F15.

*This work was done while the author was a National Academy of Sciences–National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234.

[1] Figures in brackets indicate the literature references at the end of this paper.

$$\Omega(A)=\begin{bmatrix} E_1(A) & E_3(A) & E_5(A) & \cdots & 0 \\ 1 & E_2(A) & E_4(A) & \cdots & 0 \\ 0 & E_1(A) & E_3(A) & \cdots & 0 \\ 0 & 1 & E_2(A) & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & E_n(A) \end{bmatrix}.$$

THEOREM 2: [Routh-Hurwitz] $A \in M_n(R)$ *is positive stable if and only if the leading principal minors of* $\Omega(A)$ *are positive.*

REMARK: $A \in M_n(R)$ is D-stable if and only if the leading principal minors of $\Omega(DA)$ are positive for all $D \in D_n$.

We are now in a position to give a numerical sufficient condition for 3 by 3 D-stability.

THEOREM 3: *The matrix* $A =$

$$\begin{bmatrix} x & a & b \\ \alpha & y & c \\ \beta & \gamma & z \end{bmatrix} \in M_3(R)$$

is D-stable if (*i*) $A \in P_0^+$ *and* (*ii*) $xyz > \dfrac{ac\beta + \alpha\gamma b}{2}$.

PROOF: Since the conditions (i) and (ii) are preserved under multiplication from D_3, it suffices to show that they imply stability for which we shall use theorem 2. Conditions (i) and (ii) imply the positivity of the expression:

$$(2xyz - ac\beta - \alpha\gamma b) + (x+y)(xy - a\alpha) + (x+z)(xz - b\beta) + (y+z)(yz - c\gamma)$$

This is equivalent to the inequality

$$E_1(A)E_2(A) > E_3(A).$$

Because of (i) we also have that

$$E_1(A) > 0$$

Together these mean that the leading principal minors of the 3 by 3 matrix $\Omega(A)$ are positive which completes the proof.

The conditions of theorem 3 are easily checked for a given matrix. Theoretically they are of interest in that they reveal to be D-stable a class of 3 by 3 matrices which are not known to be D-stable by any other present sufficient condition [4].

EXAMPLE: That the conditions of theorem 3 are not necessary for D-stability is shown, for instance, by letting $A =$

$$\begin{bmatrix} 6 & 5 & -1 \\ 1 & 2 & 5 \\ 5 & -3 & 1 \end{bmatrix}.$$

Then A is D-stable since $A+A^*$ is positive definite [4]. However the inequality (ii) of theorem 3 is not satisfied since $12 \not> 64$.

We end with a characterization of 4 by 4 D-stability which, unfortunately, is not numerically checkable.

THEOREM 5: $A \epsilon M_4(R)$ *is* D-*stable if and only if* (*i*) $A \epsilon P_0^+$ *and* (*ii*) *for each* $D \epsilon D_4$ *such that det* $(DA)=1$ *we have*

$$E_2(DA) > \frac{E_1(DA)}{E_3(DA)} + \frac{E_3(DA)}{E_1(DA)}.$$

PROOF: By theorem 0 we know that the D-stability of A implies $A \epsilon P_0^+$. We thus assume $A \epsilon P_0^+$ and show that A is D-stable if and only if condition (ii). However $A \epsilon M_4(R)$ is D-stable if and only if $\Omega(DA)$ has positive leading principal minors for $D \epsilon D_4$. Under the assumption $E_4(DA)=1$ which provides no loss of generality this is equivalent to $E_1(DA) > 0$, $E_2(DA)E_1(DA) > E_3(DA)$, and $E_1(DA)E_2(DA)E_3(DA) > E_1(DA)^2 + E_3(DA)^2$. The first of these conditions is subsumed in the assumption $A \epsilon P_0^+$ and the second is subsumed in the third which is equivalent to (ii). This completes the proof.

In considering sufficient conditions for or characterizations of D-stability one of course wishes conditions which are invariant under multiplication from D_n. This is a virtue of the new condition (ii) of theorem 3. Whether or not there are significant generalizations of theorem 3 is worthy of further study.

References

[1] Arrow, K. J., and McManus, M., A note on dynamic stability, Econometrica **26**, 448–454 (1958).

[2] Fiedler, M., and Ptak, V., Some generalizations of positive definiteness and monotonicity, Numerische Mathematik **9**, 163–172 (1966).

[3] Gantmacher, F. R., The Theory of Matrices V. II (Chelsea, New York, 1959).

[4] Johnson, C. R., D-stability and real and complex quadratic forms, Lin. Alg. and Applic., to appear.

[5] Quirk, J., and Ruppert, R., Qualitative economics and the stability of equilibrium, Review of Economic Studies **32**, 311–325 (1965).

(Paper 78B1–394)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 78B, No. 1, January–March 1974

Comparison of Some FORTRAN Programs for Matrix Inversion*

Kenneth E. Fitzgerald**

(August 6, 1973)

In this paper several programs for computing the inverse of a matrix are compared primarily on the basis of execution time. Accuracy estimates and two programs that use iterative refinement are included. It is shown that for small matrices, improvement procedures are worthwhile but for large matrices, one must be more careful in their use. Two other points are also brought out: the value of multiplying matrices before taking the norm of a product and the need for some kind of an error estimate to be included in the output of every program.

Key words: Error estimates; evaluation of computer programs; execution time; inverse of a matrix; iterative refinement; linear systems.

Introduction

In a previous paper [1] a number of FORTRAN programs for finding the inverse of a matrix were compared solely on the basis of accuracy. Information with respect to the execution time of these programs is also of importance and we would like to discuss this factor in this paper. This latter element becomes a very important consideration when iterative refinement is used because a more accurate inverse will clearly result but we must evaluate the added time and effort it requires.

In this paper, then, we will briefly summarize the major results of the previous work in order to have them at hand, describe the programs and the test matrices to be used, present the information with respect to execution time as well as accuracy, and then discuss the results.

Review

Any discussion of accuracy involves the concept of a norm to determine the "size" of the error and we will use two: the Frobenius norm, where $N(A) = \sqrt{\Sigma |a_{ij}|^2}$, and the maximum element norm, where $N(A) = n \cdot \max_{ij} |a_{ij}|$. It is important to note the need for the multiplication by the size of the matrix in the latter case so that the second condition for a norm, $N(AB) \leq N(A)N(B)$, will be satisfied. With regard to this particular inequality, we would like to emphasize a point made in the previous paper. The derivation of many theoretical error bounds involves the use of this property and it is clearly much simpler and faster to calculate the norm of each matrix and multiply these two numbers than to multiply two matrices and then compute the norm. However, it has been our experience that the latter procedure gives a much smaller number than the former and as a result gives a much better indication of the value of the result. This can clearly be seen by looking at columns three and four in tables 1 to 10 that follow. The error bounds used in those tables were derived in the previous paper and we will just summarize the results now.

AMS Subject Classification: 65F05, 65F35.
*An invited paper.
**Present address: Manhattan College, New York, N.Y. 10471
[1] Figures in brackets indicate the literature references at the end of this paper.

Let X be an approximate inverse and let Y be the residual matrix $I-AX$. Then for any norm, if $N(Y)<1$, we have

1. $\dfrac{N(X)}{1+N(Y)} \leq N(A^{-1}) \leq \dfrac{N(X)}{1-N(Y)}$

2. (a) $N(A^{-1}-X) \leq \dfrac{N(XY)}{1-N(Y)}$

   (b) $N(A^{-1}-X) \leq \dfrac{N(X)N(Y)}{1-N(Y)}$

3. $\dfrac{N(E)}{2N(A)} \leq N(A^{-1}-X), E=(I-XA)-(I-AX)$

4. $1-N(Y) \leq \dfrac{N(X)}{N(A^{-1})} \leq 1+N(Y)$

5. (a) $\dfrac{N(A^{-1}-X)}{N(A^{-1})} \leq \dfrac{N(XY)}{N(X)} \cdot \dfrac{1+N(Y)}{1-N(Y)}$

   (b) $\dfrac{N(A^{-1}-X)}{N(A^{-1})} \leq N(Y) \cdot \dfrac{1+N(Y)}{1-N(Y)}$

6. $\dfrac{N(E)}{2N(A)} \cdot \dfrac{1-N(Y)}{N(X)} \leq \dfrac{N(A^{-1}-X)}{N(A^{-1})}, E=(I-XA)-(I-AX)$

7. $\dfrac{N(A^{-1}-X)}{N(A^{-1})} \leq N(Y).$

The derivation of the above is based primarily on the relation $Y=I-AX$ and, after transposing and taking inverses, on the relation $(I-Y)^{-1}=I+Y+Y^2+\ldots$, if $N(Y)<1$, which also insures that the eigenvalues of Y are of modulus less than one. It will be very interesting to note in the following tables the relation between numbers 2(a), an absolute error bound and 7, a relative error bound. Before discussing the results, however, it would be good to describe the programs and test matrices.

The Computer Programs

1. LEQ

A FORTRAN subroutine used to solve the matrix equation $AX=B$ and to evaluate the determinant of A. It was written by Max Goldstein of the AEC Computing and Applied Mathematics Center at the Courant Institute of Mathematical Sciences, New York University. The Gauss elimination method is used. The matrices are normalized row-wise by dividing by the largest element of $A(I, J)$ in that row, then the A matrix is reduced to triangular form by $(N-1)$ transformations using pivotal condensation process after which X (I, J) is computed by a back-substitution process. This transforms B into X and leaves the product of the diagonal elements as the determinant of A.

2. MIDAS

A FORTRAN and ALGOL package to solve general nonsingular systems of linear algebraic equations, invert matrices, and compute determinants. Error bounds on the solution or inverse are available as an option. It was written by Peter A. Businger of Bell Telephone Laboratories, Inc., Murray Hill, New Jersey. The error bound is a bound on the distance between any element of the true inverse and the corresponding element of the computed inverse (unless the bound equals -1, in which case no bound is available). Gaussian elimination with partial pivoting is used to decompose the $N \times N$ input matrix into the product of a lower and an upper triangular matrix (LU decomposition). The magnitude of intermediate results is estimated; in case of alarming growth the program switches to complete pivoting. When solving a system of equations $Ax=b$, the accuracy of the solution obtained from the triangular system is improved by iteration; in the case of matrix inversion the iteration is omitted for the sake of computational efficiency. (We note that the error bound is essentially $N(X)N(Y)/[1-N(Y)]$, $Y=I-AX$.)

3. MINV

A FORTRAN subroutine, one of the IBM System/360 Scientific Subroutine Package. It inverts a general matrix by the standard Gauss-Jordan method. The determinant is also calculated. A determinant of zero indicates a singular matrix.

4. SPINV

A single precision FORTRAN IV program for inverting a matrix or solving a set of linear equations. To a program from the SHARE library (7090–F1 3180INV1 Single Precision Matrix Inversion with Selective Pivoting, written by A. R. Sadaka), Sally T. Peavy, National Bureau of Standards, incorporated accuracy checks. This is also the routine used by INVERT of OMNITAB.

5. SOLVE

A FORTRAN program by Cleve Moler given in the book *Computer Solution of Linear Algebraic Systems* by George Forsythe and Cleve Moler. It uses Gauss elimination with partial pivoting and has a subroutine IMPRUV, which can be called to improve the solution of a linear algebraic system. Appropriate messages for various kinds of singularity are available. It is presently undergoing some changes to increase efficiency in most FORTRAN systems although these changes should not materially alter the numerical behavior [2].

6. LINEQ1

A FORTRAN subroutine used to solve the real matrix equation $AX=B$ and written by David S. Dodson, Department of Computer Sciences, Purdue University. The matrix A is factored into lower and upper triangular matrices L and U and then the equations $LZ=B$ and $UX=Z$ are solved in turn. Double precision accumulation of inner products and iterative refinement are used to improve accuracy.

The Test Matrices

An indication of the degree of difficulty that may be encountered in computing the inverse of a matrix is given by the "condition number" of a matrix. There are many ways of arriving at such a number and the one we shall use is called the P-condition number:

$$P(A)=\left|\frac{\lambda}{\mu}\right|$$

536-390 O - 74 - 2

where λ is an eigenvalue of largest modulus and μ, of smallest. We will give this value for each of our test matrices. Since the execution time for inverting matrices of small size (20 by 20) is rather minimal, roughly two seconds, we used matrices of order 100. This means, of course, that the Hilbert matrix had to be excluded as a test matrix ($\log_e P(H_n) \approx 3.5n$, $n=100$) and hence we used only the following five matrices.

1. A_{100}

This is a 100×100 matrix where $A_k = (1/k)I + J$ and J is the 100×100 matrix of all ones. $P(A_k) = 1 + 100k$. The integer form for use as a test matrix is obviously achieved upon multiplication by k.

$$(kA_k)^{-1} = I - \frac{k}{1+100k} \quad J$$

$$P(A_{100}) = 10001$$

$$\approx .10E+05$$

2. A_{1000}

The same as above except that we change the value of k.

$$P(A_{1000}) = 100001$$

$$\approx .10+06$$

3. A_{10000}

The same as above except that we change the value of k.

$$P(A_{10000}) = 1000001$$

$$\approx .10E+07$$

4. T_{100}

This is a 100×100 tridiagonal matrix with -2 on the diagonal, 1 above and below the diagonal, and 0 elsewhere.

$$T = (t_{ij}),\ t_{ij} = \begin{cases} -2, & i=j \\ 1, & |i-j|=1 \\ 0, & |i-j| \geq 2. \end{cases}$$

$$P(T_n) \approx (4/\pi^2)n^2$$

$$\approx .41E+04$$

5. T_{100}^2

This is just the square of the above matrix.

$$P(T_{100}^2) \approx (4/\pi^2)^2\,(n^2)^2$$

$$\approx .16E+08$$

The Results

In tables 1 to 10 we give the information with regard to accuracy, using the two different norms, and in tables 11 to 22, the information concerning time; the last set of tables, 23 to 32, will combine the two pieces of information. We will then discuss the results.

TABLE 1. *Accuracy of results*

Program		Max elt of check vector*	$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	$\frac{N(X)N(Y)}{1-N(Y)}$	Relative error	$N(I-XA)$	Rank
LEQ		.68 E−05	.18 E−02	.45 E−05	.18 E−01	.45 E−06	.30 E−02	6
MIDAS		.19 E−03	.24 E−01	.39 E−05	.25 E+00	.41 E−06	.12 E−02	4
MINV		.60 E−06	.79 E−03	.12 E−06	.79 E−02	.12 E−07	.46 E−03	3
SPINV		.79 E−05	.62 E−02	.70 E−05	.61 E−01	.71 E−06	.61 E−02	7
SOLVE	No IMPRUV	.13 E−05	.16 E−02	.42 E−05	.16 E−01	.43 E−06	.33 E−02	5
	IMPRUV	.86 E−07	.22 E−04	.69 E−07	.22 E−03	.70 E−08	.26 E−04	2
LINEQ1		.15 E−06	.25 E−04	.46 E−07	.25 E−03	.47 E−08	.32 E−04	1

Matrix = A_{100}
Condition Number = .10 E+05
Norm = FROBENIUS

*Difference from 1.

X = Approximate inverse

$Y = I - AX$

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)} \leq \frac{N(X)N(Y)}{1-N(Y)}$$

$$\frac{N(A^{-1}-X)}{N(A^{-1})} \leq \frac{N(XY)}{1-N(Y)} \cdot \frac{1+N(Y)}{N(X)}$$

TABLE 2. *Accuracy of results*

Program		Max elt of check vector*	$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	$\frac{N(X)N(Y)}{1-N(Y)}$	Relative error	$N(I-XA)$	Rank
LEQ		.11 E−03	.20 E−01	.55 E−04	.20 E+00	.56 E−05	.20 E−01	7
MIDAS		.15 E−02	.20 E+00	.11 E−04	.25 E+01	.13 E−05	.19 E−01	5
MINV		.41 E−05	.40 E−01	.41 E−06	.42 E+00	.43 E−07	.39 E−01	3
SPINV		.44 E−03	.62 E−01	.11 E−04	.65 E+00	.11 E−05	.62 E−01	5
SOLVE	No IMPRUV	.69 E−05	.16 E−01	.83 E−05	.16 E+00	.85 E−06	.24 E−01	4
	IMPRUV	.88 E−07	.41 E−03	.13 E−07	.41 E−02	.13 E−08	.47 E−03	1
LINEQ1		.15 E−06	.33 E−03	.32 E−07	.32 E−02	.32 E−08	.33 E−03	2

Matrix = A_{1000}
Condition Number = .10 E+06
Norm = FROBENIUS

*Difference from 1.

X = Approximate inverse

$Y = I - AX$

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)} \leq \frac{N(X)N(Y)}{1-N(Y)}$$

$$\frac{N(A^{-1}-X)}{N(A^{-1})} \leq \frac{N(XY)}{1-N(Y)} \cdot \frac{1+N(Y)}{N(X)}$$

TABLE 3. *Accuracy of results*

Program		Max elt of check vector*	$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	$\frac{N(X)N(Y)}{1-N(Y)}$	Relative error	$N(I-XA)$	Rank
LEQ		.23 E−02	.19 E+00	.13 E−02	.23 E+01	.16 E−03	.21 E+00	6
MIDAS		.12 E−01	.46 E+01				.18 E+00	
MINV		.45 E−05	.45 E+00	.46 E−06	.81 E+01	.68 E−07	.44 E+00	3
SPINV		.34 E−05	.61 E+00	.26 E−03	.16 E+02	.42 E−04	.61 E+00	5
SOLVE	No IMPRUV	.38 E−05	.16 E+00	.12 E−03	.19 E+01	.14 E−04	.24 E+00	4
	IMPRUV	.00	.33 E−02	.76 E−08	.33 E−01	.77 E−09	.54 E−02	1
LINEQL		.20 E−06	.11 E−01	.19 E−07	.11 E+00	.19 E−08	.11 E−01	2

Matrix = A_{10000}
Condition Number = .10 E+07
Norm = FROBENIUS

*Difference from 1.

X = Approximate inverse

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)} \leq \frac{N(X)N(Y)}{1-N(Y)}$$

$Y = I-AX$

$$\frac{N(A^{-1}-X)}{N(A^{-1})} \leq \frac{N(XY)}{1-N(Y)} \cdot \frac{1+N(Y)}{N(X)}$$

TABLE 4. *Accuracy of results*

Program		Max elt of check vector*	$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	$\frac{N(X)N(Y)}{1-N(Y)}$	Relative error	$N(I-XA)$	Rank
LEQ		.14 E−03	.19 E−04	.16 E−01	.21 E−01	.15 E−04	.57 E−04	4
MIDAS		.14 E−03	.29 E−04	.16 E−01	.31 E−01	.15 E−04	.28 E−04	4
MINV		.75 E−05	.18 E−04	.69 E−03	.19 E−01	.64 E−06	.16 E−04	3
SPINV		.14 E−03	.55 E−04	.16 E−01	.59 E−01	.15 E−04	.52 E−04	4
SOLVE	No IMPRUV	.10 E−04	.19 E−04	.16 E−01	.21 E−01	.15 E−04	.57 E−04	4
	IMPRUV	.15 E−06	.86 E−05	.69 E−05	.93 E−02	.65 E−08	.86 E−05	1
LINEQ1		.15 E−06	.86 E−05	.69 E−05	.93 E−02	.65 E−08	.86 E−05	1

Matrix = T_{100}
Condition Number = .41 E+04
Norm = FROBENIUS

*Difference from 1.

X = Approximate inverse

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)} \leq \frac{N(X)N(Y)}{1-N(Y)}$$

$Y = I-AX$

$$\frac{N(A^{-1}-X)}{N(A^{-1})} \leq \frac{N(XY)}{1-N(Y)} \cdot \frac{1+N(Y)}{N(X)}$$

TABLE 5. *Accuracy of results*

Program		Max elt of check vector*	$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	$\frac{N(X)N(Y)}{1-N(Y)}$	Relative error	$N(I-XA)$	Rank
LEQ		.60 E+00	.83 E−01	.82 E+05	.10 E+06	.78 E−01	.20 E+03	6
MIDAS		.29 E+00	.10 E+00	.39 E+05	.12 E+06	.39 E−01	.74 E+02	5
MINV		.21 E+00	.15 E+00	.27 E+05	.18 E+06	.30 E−01	.16 E+00	3
SPINV		.26 E+00	.27 E+03				.24 E+00	
SOLVE	No IMPRUV	.14 E−01	.13 E+00	.36 E+05	.17 E+06	.37 E−01	.18 E+03	4
	IMPRUV	.00	.30 E−01	.70 E−02	.34 E+05	.67 E−08	.30 E−01	1
LINEQ1		.31 E−05	.29 E−01	.43 E+00	.32 E+05	.41 E−06	.30 E−01	2

Matrix = T^2_{100}
Condition Number = .16 E+08
Norm = FROBENIUS

*Difference from 1.

X = Approximate inverse

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)} \leq \frac{N(X)N(Y)}{1-N(Y)}$$

$Y = I-AX$

$$\frac{N(A^{-1}-X)}{N(A^{-1})} \leq \frac{N(XY)}{1-N(Y)} \cdot \frac{1+N(Y)}{N(X)}$$

TABLE 6. *Accuracy of results*

Program		Max elt of check vector*	$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	$\frac{N(X)N(Y)}{1-N(Y)}$	Relative error	$N(I-XA)$	Rank
LEQ		.68 E−05	.31 E−02	.67 E−04	.30 E+00	.68 E−06	.21 E−01	4
MIDAS		.19 E−03	.24 E+00	.17 E−03	.31 E+02	.21 E−05	.33 E−02	7
MINV		.60 E−06	.69 E−02	.81 E−06	.69 E+00	.82 E−08	.55 E−03	3
SPINV		.79 E−05	.92 E−02	.13 E−03	.92 E+00	.13 E−05	.89 E−02	6
SOLVE	No IMPRUV	.13 E−05	.25 E−02	.84 E−04	.24 E+00	.85 E−06	.26 E−01	5
	IMPRUV	.86 E−07	.46 E−04	.69 E−06	.45 E−02	.70 E−08	.49 E−04	1
LINEQ1		.15 E−06	.69 E−04	.69 E−06	.68 E−02	.70 E−08	.79 E−04	1

Matrix = A_{100}
Condition Number = .10 E+05
Norm = $n \cdot$ (Maximum Element)

*Difference from 1.

X = Approximate inverse

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)} \leq \frac{N(X)N(Y)}{1-N(Y)}$$

$Y = I-AX$

$$\frac{N(A^{-1}-X)}{N(A^{-1})} \leq \frac{N(XY)}{1-N(Y)} \cdot \frac{1+N(Y)}{N(X)}$$

TABLE 7. *Accuracy of results*

Program		Max elt of check vector*	$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	$\frac{N(X)N(Y)}{1-N(Y)}$	Relative error	$N(I-XA)$	Rank
LEQ		.11 E−03	.37 E−01	.58 E−03	.38 E+01	.61 E−05	.79 E−01	4
MIDAS		.15 E−02	.20 E+01				.31 E−01	
MINV		.40 E−05	.16 E+00	.18 E−05	.19 E+02	.21 E−07	.39 E−01	3
SPINV		.44 E−03	.92 E−01	.81 E−03	.10 E+02	.90 E−05	.92 E−01	6
SOLVE	No IMPRUV	.69 E−05	.29 E−01	.74 E−03	.29 E+01	.77 E−05	.12 E+00	5
	IMPRUV	.88 E−07	.62 E−03	.11 E−06	.62 E−01	.11 E−08	.70 E−03	1
LINEQ1		.15 E−06	.97 E−03	.64 E−06	.96 E−01	.65 E−08	.97 E−03	2

Matrix = A_{1000}
Condition Number = .10 E+06
Norm = $n \cdot$ (Maximum Element)

*Difference from 1.

X = Approximate inverse

$Y = I - AX$

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)} \leq \frac{N(X)N(Y)}{1-N(Y)}$$

$$\frac{N(A^{-1}-X)}{N(A^{-1})} \leq \frac{N(XY)}{1-N(Y)} \cdot \frac{1+N(Y)}{N(X)}$$

TABLE 8. *Accuracy of results*

Program		Max elt of check vector*	$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	$\frac{N(X)N(Y)}{1-N(Y)}$	Relative error	$N(I-XA)$	Rank
LEQ		.23 E−02	.30 E+00	.15 E−01	.43 E+02	.20 E−03	.76 E+00	4
MIDAS		.12 E−01	.46 E+02				.31 E+00	
MINV		.45 E−05	.15 E+01				.44 E+00	
SPINV		.34 E−05	.86 E+00	.71 E−01	.62 E+03	.13 E−02	.88 E+00	5
SOLVE	No IMPRUV	.38 E−05	.22 E+00	.13 E−01	.27 E+02	.15 E−03	.16 E+01	3
	IMPRUV	.00	.50 E−02	.47 E−07	.50 E+00	.47 E−09	.11 E−01	1
LINEQ1		.20 E−06	.12 E−01	.80 E−06	.12 E+01	.82 E−08	.12 E−01	2

Matrix = A_{10000}
Condition Number = .10 E+07
Norm = $n \cdot$ (Maximum Element)

*Difference from 1.

X = Approximate inverse

$Y = I - AX$

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)} \leq \frac{N(X)N(Y)}{1-N(Y)}$$

$$\frac{N(A^{-1}-X)}{N(A^{-1})} \leq \frac{N(XY)}{1-N(Y)} \cdot \frac{1+N(Y)}{N(X)}$$

TABLE 9. *Accuracy of results*

Program		Max elt of check vector*	$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	$\frac{N(X)N(Y)}{1-N(Y)}$	Relative error	$N(I-XA)$	Rank
LEQ		.14 E−03	.12 E−03	.33 E−01	.30 E+00	.13 E−04	.36 E−03	4
MIDAS		.14 E−03	.14 E−03	.32 E−01	.36 E+00	.13 E−04	.13 E−03	4
MINV		.75 E−05	.12 E−03	.14 E−02	.30 E+00	.55 E−06	.72 E−04	3
SPINV		.14 E−03	.33 E−03	.33 E−01	.84 E+00	.13 E−04	.36 E−03	4
SOLVE	No IMPRUV	.10 E−04	.12 E−03	.33 E−01	.30 E+00	.13 E−04	.36 E−03	4
	IMPRUV	.15 E−06	.36 E−04	.24 E−04	.90 E−01	.94 E−08	.36 E−04	1
LINEQ1		.15 E−06	.36 E−04	.24 E−04	.90 E−01	.94 E−08	.36 E−04	1

Matrix = T_{100}
Condition Number = .41 E+04
Norm = $n \cdot$ (Maximum Element)

*Difference from 1.

X = Approximate inverse

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)} \leq \frac{N(X)N(Y)}{1-N(Y)}$$

$Y = I-AX$

$$\frac{N(A^{-1}-X)}{N(A^{-1})} \leq \frac{N(XY)}{1-N(Y)} \cdot \frac{1+N(Y)}{N(X)}$$

TABLE 10. *Accuracy of results*

Program		Max elt of check vector*	$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	$\frac{N(X)N(Y)}{1-N(Y)}$	Relative error	$N(I-XA)$	Rank
LEQ		.60 E+00	.88 E+00	.12 E+07	.17 E+08	.10 E+01	.11 E+04	5
MIDAS		.29 E+00	.71 E+00	.24 E+06	.54 E+07	.18 E+00	.48 E+03	4
MINV		.21 E+00	.76 E+00	.19 E+06	.65 E+07	.16 E+00	.93 E+00	3
SPINV		.26 E+00	.14 E+04				.12 E+01	
SOLVE	No IMPRUV	.14 E−01	.22 E+01				.13 E+04	
	IMPRUV	.00	.17 E+00	.29 E−01	.44 E+06	.16 E−07	.17 E+00	1
LINEQ1		.31 E−05	.15 E+00	.93 E+00	.37 E+06	.53 E−06	.17 E+00	2

Matrix = T^2_{100}
Condition Number = .16 E+08
Norm = $n \cdot$ (Maximum Element)

*Difference from 1.

X = Approximate inverse

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)} \leq \frac{N(X)N(Y)}{1-N(Y)}$$

$Y = I-AX$

$$\frac{N(A^{-1}-X)}{N(A^{-1})} \leq \frac{N(XY)}{1-N(Y)} \cdot \frac{1+N(Y)}{N(X)}$$

NOTE: In tables 11 to 17, the increase in time for calculating the error bounds for T^2_{100} is due to the fact that the exact inverse was also calculated and used in evaluating the results for that matrix. Although these figures are not listed in this paper, they once again show the value of using $N(XY)$ instead of $N(X)N(Y)$.

TABLE 11. *Breakdown of run time in seconds per program*

Matrix	Set up	Solution	Error bounds	Total
A_{100}	6.4	17.6	59.4	83.4
A_{1000}	6.4	17.4	58.7	82.6
A_{10000}	6.4	17.6	58.6	82.6
T_{100}	3.8	17.4	58.6	79.9
T^2_{100}	18.1	17.6	85.7	121.4

Program = LEQ

TABLE 12. *Breakdown of run time in seconds per program*

Matrix	Set up	Solution	Error bounds	Total
A_{100}	6.3	49.1	59.6	108.0
A_{1000}	6.3	41.7	59.0	107.0
A_{10000}	6.2	42.6	60.2	109.0
T_{100}	3.8	42.3	59.7	105.8
T^2_{100}	18.3	42.6	86.5	147.5

Program = MIDAS

TABLE 13. *Breakdown of run time in seconds per program*

Matrix	Set up	Solution	Error bounds	Total
A_{100}	8.4	29.3	61.1	98.7
A_{1000}	6.4	29.4	59.7	95.5
A_{10000}	6.5	29.4	59.6	95.5
T_{100}	3.8	29.9	58.8	92.5
T^2_{100}	18.3	30.0	86.6	134.8

Program = MINV

TABLE 14. *Breakdown of run time in seconds per program*

Matrix	Set up	Solution	Error bounds	Total
A_{100}	6.3	42.1	59.3	107.7
A_{1000}	6.4	42.0	59.2	107.6
A_{10000}	6.3	42.1	59.4	107.7
T_{100}	3.7	41.9	58.5	104.0
T^2_{100}	18.2	42.7	85.5	146.3

Program = SPINV

TABLE 15. *Breakdown of run time in seconds per program*

Matrix	Set up	Solution	Error bounds	Total
A_{100}	6.3	10.7	59.8	76.8
A_{1000}	6.3	10.6	59.7	76.7
A_{10000}	6.2	10.4	58.9	75.6
T_{100}	3.7	10.4	52.9	67.1
T^2_{100}	18.2	10.6	87.1	115.9

Program = SOLVE (No IMPRUV)

TABLE 16. *Breakdown of run time in seconds per program*

Matrix	Set up	Solution	Error bounds	Total
A_{100}	6.4	55.7	60.1	122.2
A_{1000}	6.3	55.6	59.1	121.0
A_{10000}	6.2	54.6	58.9	119.8
T_{100}	3.8	55.6	58.5	117.8
T^2_{100}	14.9	136.7	84.8	236.4

Program = SOLVE (With IMPRUV)

TABLE 17. *Breakdown of run time in seconds per program*

Matrix	Set up	Solution	Error bounds	Total
A_{100}	6.2	49.0	59.6	114.9
A_{1000}	6.4	49.2	59.6	115.2
A_{10000}	6.4	49.8	60.1	116.3
T_{100}	3.8	49.6	57.1	110.5
T^2_{100}	18.3	79.8	84.9	183.0

Program = LINEQ1

TABLE 18. *Comparison of programs according to time in seconds*

Program		Set up	Solution	Error bounds	Total	Rank
LEQ		6.4	17.6	59.4	83.4	2
MIDAS		6.3	42.1	59.6	108.0	4
MINV		8.4	29.3	61.1	98.7	3
SPINV		6.3	42.1	59.3	107.7	4
SOLVE	No IMPRUV	6.3	10.7	59.8	76.8	1
	IMPRUV	6.4	55.7	60.1	122.2	7
LINEQ1		6.2	49.0	59.6	114.9	6

Matrix = A_{100} Condition Number = .10E + 04

TABLE 19. *Comparison of programs according to time in seconds*

Program		Set up	Solution	Error bounds	Total	Rank
LEQ		6.4	17.4	58.7	82.6	2
MIDAS		6.3	41.7	59.0	107.0	4
MINV		6.4	29.4	59.7	95.5	3
SPINV		6.4	42.0	59.2	107.6	5
SOLVE	No IMPRUV	6.3	10.6	59.7	76.7	1
	IMPRUV	6.3	55.6	59.1	121.0	7
LINEQ1		6.4	49.2	59.6	115.2	6

Matrix = A_{1000} Condition Number = .10 E + 05

TABLE 20. *Comparison of programs according to time in seconds*

Program		Set up	Solution	Error bounds	Total	Rank
LEQ		6.4	17.6	58.6	82.6	2
MIDAS		6.2	42.6	60.2	109.0	5
MINV		6.5	29.4	59.6	95.5	3
SPINV		6.3	42.1	59.4	107.7	4
SOLVE	No IMPRUV	6.2	10.4	58.9	75.6	1
	IMPRUV	6.2	54.6	58.9	119.8	7
LINEQ1		6.4	49.8	60.1	116.3	6

Matrix = A_{10000} Condition Number = .10 E + 06

TABLE 21. *Comparison of programs according to time in seconds*

Program		Set up	Solution	Error bounds	Total	Rank
LEQ		3.8	17.4	58.6	79.9	2
MIDAS		3.8	42.3	59.7	105.8	5
MINV		3.8	29.9	58.8	92.5	3
SPINV		3.7	41.9	58.5	104.0	4
SOLVE	No IMPRUV	3.7	10.4	52.9	67.1	1
	IMPRUV	3.8	55.6	58.5	117.8	7
LINEQ1		3.8	49.6	57.1	110.5	6

Matrix = T_{100} Condition Number = .41 E + 05

TABLE 22. *Comparison of programs according to time in seconds*

Program		Set up	Solution	Error bounds	Total	Rank
LEQ		18.1	17.6	85.7	121.4	2
MIDAS		18.3	42.6	86.5	147.5	4
MINV		18.3	30.0	86.6	134.8	3
SPINV		18.2	42.7	N(Y) > 1		
SOLVE	No IMPRUV	18.2	10.6	87.1	115.9	1
	IMPRUV	14.9	136.7	84.8	236.4	6
LINEQ1		18.3	79.8	84.9	183.0	5

Matrix = T^2_{100} Condition Number = .16 E + 08

TABLE 23. *Summary of results*

Program		$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	Subroutine time (Seconds)
LEQ		.18 E − 02	.45 E − 05	17.6
MIDAS		.24 E − 01	.39 E − 05	42.1
MINV		.79 E − 03	.12 E − 06	29.3
SPINV		.62 E − 02	.70 E − 05	42.1
SOLVE	No IMPRUV	.16 E − 02	.42 E − 05	10.7
	IMPRUV	.22 E − 04	.69 E − 07	55.7
LINEQ1		.25 E − 04	.46 E − 07	49.0

Matrix = A_{100}
Condition Number = .10 E + 04
Norm = FROBENIUS

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)}$$

TABLE 24. *Summary of results*

Program		$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	Subroutine time (Seconds)
LEQ		.20 E−01	.55 E−04	17.4
MIDAS		.20 E+00	.11 E−04	41.7
MINV		.40 E−01	.41 E−06	29.4
SPINV		.62 E−01	.11 E−04	42.0
SOLVE	No IMPRUV	.16 E−01	.83 E−05	10.6
	IMPRUV	.41 E−03	.13 E−07	55.6
LINEQ1		.33 E−03	.32 E−07	49.2

Matrix = A_{1000}
Condition Number = .10 E+05
Norm = FROBENIUS

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)}$$

TABLE 25. *Summary of results*

Program		$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	Subroutine time (Seconds)
LEQ		.19 E+00	.13 E−02	17.6
MIDAS		.46 E+01		
MINV		.45 E+00	.46 E−06	29.4
SPINV		.61 E+00	.26 E−03	42.1
SOLVE	No IMPRUV	.16 E+00	.12 E−03	10.4
	IMPRUV	.33 E−02	.76 E−08	54.6
LINEQ1		.11 E−01	.19 E−07	49.8

Matrix = A_{10000}
Condition Number = .10 E+06
Norm = FROBENIUS

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)}$$

TABLE 26. *Summary of results*

Program		$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	Subroutine time (Seconds)
LEQ		.19 E−04	.16 E−01	17.4
MIDAS		.29 E−04	.16 E−01	42.3
MINV		.18 E−04	.69 E−03	29.9
SPINV		.55 E−04	.16 E−01	41.9
SOLVE	No IMPRUV	.19 E−04	.16 E−01	10.4
	IMPRUV	.86 E−05	.69 E−05	55.6
LINEQ1		.86 E−05	.69 E−05	49.6

Matrix = T_{100}
Condition Number = .41 E+05
Norm = FROBENIUS

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)}$$

TABLE 27. *Summary of results*

Program		$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	Subroutine time (Seconds)
LEQ		.83 E−01	.82 E+05	17.6
MIDAS		.10 E+00	.39 E+05	42.6
MINV		.15 E+00	.27 E+05	30.0
SPINV		.27 E+03		
SOLVE	No IMPRUV.	.13 E+00	.36 E+05	10.6
	IMPRUV	.30 E−01	.70 E−02	136.7
LINEQ1		.29 E−01	.43 E+00	79.8

Matrix = T^2_{100}
Condition Number = .16 E+08
Norm = FROBENIUS

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)}$$

TABLE 28. *Summary of results*

Program		$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	Subroutine time (Seconds)
LEQ		.31 E−02	.67 E−04	17.6
MIDAS		.24 E+00	.17 E−03	42.1
MINV		.69 E−02	.81 E−06	29.3
SPINV		.92 E−02	.13 E−03	42.1
SOLVE	No IMPRUV.	.25 E−02	.84 E−04	10.7
	IMPRUV	.46 E−04	.69 E−06	55.7
LINEQ1		.69 E−04	.69 E−06	49.0

Matrix = A_{100}
Condition Number = .10 E+04
Norm = $n \cdot$ (Maximum Element)

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)}$$

TABLE 29. *Summary of results*

Program		$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	Subroutine time (Seconds)
LEQ		.37 E−01	.58 E−03	17.4
MIDAS		.20 E+01		
MINV		.16 E+00	.18 E−05	29.4
SPINV		.92 E−01	.81 E−03	42.0
SOLVE	No IMPRUV	.29 E−01	.74 E−03	10.6
	IMPRUV	.62 E−03	.11 E−06	55.6
LINEQ1		.97 E−03	.64 E−06	49.2

Matrix = A_{1000}
Condition Number = .10 E+05
Norm = $n \cdot$ (Maximum Element)

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)}$$

TABLE 30. *Summary of results*

Program		$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	Subroutine time (Seconds)
LEQ		.30 E+00	.15 E−01	17.6
MIDAS		.46 E+02		
MINV		.15 E+01		
SPINV		.86 E+00	.71 E−01	42.1
SOLVE	No IMPRUV	.22 E+00	.13 E−01	10.4
	IMPRUV	.50 E−02	.47 E−07	54.6
LINEQ1		.12 E−01	.80 E−06	49.8

Matrix = A_{10000}
Condition Number = .10 E+06
Norm = $n \cdot$ (Maximum Element)

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)}$$

TABLE 31. *Summary of results*

Program		$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	Subroutine time (Seconds)
LEQ		.12 E−03	.33 E−01	17.4
MIDAS		.14 E−03	.32 E−01	42.3
MINV		.12 E−03	.14 E−02	29.9
SPINV		.33 E−03	.33 E−01	41.9
SOLVE	No IMPRUV	.12 E−03	.33 E−01	10.4
	IMPRUV	.36 E−04	.24 E−04	55.6
LINEQ1		.36 E−04	.24 E−04	49.6

Matrix $= T_{100}$
Condition Number = .41 E+05
Norm = $n \cdot$ (Maximum Element)

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)}$$

TABLE 32. *Summary of results*

Program		$N(I-AX)$	$\frac{N(XY)}{1-N(Y)}$	Subroutine time (Seconds)
LEQ		.88 E+00	.12 E+07	17.6
MIDAS		.71 E+00	.24 E+06	42.6
MINV		.76 E+00	.19 E+06	30.0
SPINV		.14 E+04		
SOLVE	No IMPRUV	.22 E+01		
	IMPRUV	.17 E+00	.29 E−01	136.7
LINEQ1		.15 E+00	.93 E+00	79.8

Matrix $= T^3_{100}$
Condition Number = .16 E+08
Norm = $n \cdot$ (Maximum Element)

$$N(A^{-1}-X) \leq \frac{N(XY)}{1-N(Y)}$$

Discussion

As was mentioned earlier, the value of multiplying matrices before taking the norm of a product of two matrices is clearly demonstrated in tables 1 through 10. $N(I-AX)$ is a relative error bound and $N(XY)/[1-N(Y)]$ is an absolute error bound and yet for the A_k matrices, the latter was always smaller than the former. For the T matrices this is not true except when iterative refinement (IMPRUV) is used. In these cases, however, the relative error in column 5 is a much better bound than $N(I-AX)$. As was indicated in the previous paper also, this absolute error bound is as close to the actual error as one could expect.

Let us now turn our attention to the time element. For small matrices, the use of iterative refinement added such a small increase (1 second for the 20×20 case) that it seems definitely useful. For larger matrices, however, the picture is not quite so clear, but let us make some general observations first.

As will be noticed in tables 11 to 22, the information that took longest to gather was the error bounds. It is not necessary to calculate all this information in a particular run but only what would be useful. What is included in that part of the program is the computation of $I-AX, I-XA, X(I-AX)$, the difference between the residuals and the calculation of the two norms for these quantities. It is up to the user to decide what is the necessary information.

From tables 1 to 10 it can be seen that the programs without iterative refinement performed quite similarly concerning accuracy, with MINV consistently being slightly better. Iterative refinement, of course, had its desirable effect. From tables 11 to 17 we see that each program was consistent in its execution time for the different matrices with the exception of the two programs that used iterative refinement, LINEQ1 and SOLVE with IMPRUV. In these programs more iterations were needed for the most ill-conditioned matrix, T^2_{100}.

In tables 18 to 22 it can be seen that SOLVE without IMPRUV was definitely the fastest. As a point of information for those familiar with this program, we used the new version given in (2) to find the inverse of T_{100}. The times are given in the following table:

	Old version	New version
DECOMP	2.7	.3
SOLVE	8.2	8.7
Total	10.9	9.0

Admittedly, the times are rather minimal but the decrease for DECOMP is considerable. Whether this reduction is primarily due to omitting the scaling used in the old version or to the different way of writing matrix multiplication is not clear. However, abundant support for the latter is given in (2) and this increase in efficiency makes SOLVE without IMPRUV even faster than the other programs. We might add that the numerical accuracy did not change: all digits were identical in both runs.

The two programs using iterative refinement were quite comparable except in the case of T^2_{100}. For this matrix, LINEQ1 had 6 digit accuracy whereas SOLVE with IMPRUV had 8 digit accuracy in almost every element.

The more important question of whether iterative refinement is worth the extra time remains. This is an almost impossible question to answer in the abstract. The proposer of the problem is really the only one who can make that decision. If an accurate inverse in itself is the desired end-product, then some criterion for $N(A^{-1}-X)$ may be used to decide. (It is certainly important that this criterion be included in the output of *every* program anyway.) It seems from tables 23 to 35 that for A_{100}, A_{1000}, and A_{10000} iterative refinement would not be needed and that SOLVE without IMPRUV would be the most efficient—an excellent error bound in the fastest time. It would seem that for T^2_{100} some improvement *is* necessary. However, to let the program run its full length might not be necessary. From our experience with SOLVE with IMPRUV using a UNIVAC 1108, we estimate for this size matrix approximately 20–25 seconds per iteration and each iteration yields at least one digit improvement. The way SOLVE is set up allows the user to decide whether or not to use the subroutine IMPRUV and, if used, the maximum number of iterations to be performed. Approximation of the number of correct digits in the nonimproved computed inverse is also available from one iteration in this subroutine. (See [3], p. 50.) We have found those estimates to be very good.

In summary, for programs without iterative refinement, it seems that SOLVE is the fastest and MINV, the most accurate. It also seems that SOLVE has the best combination of accuracy and time. If iterative refinement may be desired, it seems the optional nature of IMPRUV and its added information would indicate its high value.

It still remains up to the originator of the problem to decide just exactly what is desired. At any rate, the purpose of the information contained herein is to help whomever has to make the decision.

References

[1] Fitzgerald, Kenneth E., Error estimates for the solution of linear algebraic systems, J. Res. Nat. Bur. Stand. (U.S.) **74B** (Math. Sci.), No. 4, 251–310 (Oct.–Dec. 1970).

[2] Moler, Cleve B., Matrix computations with FORTRAN and paging, Comm. ACM **15**, 4 (April 1972), 268–270, 274.

[3] Forsythe, George, and Moler, Cleve B., Computer Solution of Linear Algebraic Systems (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967).

(Paper 78B1–395)

536-300 O - 74 - 3

JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematical Sciences
Vol. 78B, No. 1, January–March 1974

On Characters of Subgroups*

Russell Merris**

(October 16, 1973)

Let H be a subgroup of G. Let χ be an irreducible character of H. Let χ^G be the character of G induced by χ. The irreducibility of χ^G is discussed. In particular, if H is normal in G, then χ^G is irreducible if and only if χ cannot be extended to any subgroup of G which properly contains H.

These results have application to the determination of irreducibility of a class of representations of the full linear groups.

Key words: Frobenius Reciprocity Theorem.

I. First Introduction

Let H be a subgroup of G. Let χ be an irreducible complex character of H. In the course of the author's study of a class of representations of the full linear group, the following criterion arose:

Condition 1. There is an irreducible character λ of G such that $(\lambda, \chi)_H = 1$, and $(\xi, \chi)_H = 0$, for every irreducible character ξ of G different from λ. (Here, of course,

$$(\xi, \chi)_H = \frac{1}{o(H)} \sum_{h \in H} \xi(h)\, \chi(h^{-1}).$$

It turns out that when $G = S_m$, the symmetric group, Condition 1 is equivalent to the irreducibility of a certain representation of the full linear group [3].[1] The main purpose of this (essentially expository) note is to investigate character theoretic statements which are related to Condition 1.

II. Second Introduction

Let H be a subgroup of G and let χ be an irreducible character of H. Can we obtain from this situation any information about the irreducible characters of G? It would be most pleasant, for example, if χ could be extended to a character of G. But, this is not always possible.

One general method to obtain a character of G from χ goes as follows: Define χ^* on G by $\chi^*(h) = \chi(h)$ for $h \in H$, and $\chi^*(g) = 0$ for $g \in G \setminus H$. then

$$\chi^G(g) = \frac{1}{o(H)} \sum_{f \in G} \chi^*(f^{-1} g f), \qquad g \in G,$$

turns out to be a character of G whose degree is $\chi(id)\,[G:H]$. It is called the character of G *induced* by χ [1, 2].

AMS Subject Classification: Primary 20C15; Secondary 15A69, 20G05.
*An invited paper.
**Present address: Instituto de Física e Matemática, Av. Gama Pinto, 2, Lisbon, 4 (Portugal).
[1] Figures in brackets indicate the literature references at the end of this paper.

Now, of course, we would like to know something about χ^G. For example, is it an irreducible character? In general, the answer is no. We are indebted to Frobenius for the following very useful result:

(Frobenius Reciprocity) Theorem: Let H be a subgroup G. Let χ and λ be characters of H and G respectively. Then

$$(\chi, \lambda)_H = (\chi^G, \lambda)_G.$$

As we shall see, the irreducibility of χ^G is related to the extendability of χ.

III. Results

Suppose $g \epsilon G$. We let χ^g denote the character of gHg^{-1} defined by

$$\chi^g(gHg^{-1}) = \chi(h), \qquad h \epsilon H.$$

THEOREM 1: Let H be a subgroup of G. Let χ be an irreducible character of H. The following are equivalent

a. Condition 1.
b. χ^G is irreducible (in fact $\chi^G = \lambda$).
c. For all $g \epsilon G \setminus H$, χ^g and χ are different characters of $H \cap gHg^{-1}$.

THEOREM 2: If χ^G is irreducible, then χ cannot be extended to any subgroup of G which properly contains H.

Unfortunately, the converse of Theorem 2 is not true in general. For example, let $G = S_4$. Let H be the subgroup generated by $\{(14)(23), (1234)\}$. (Then H is the dihedral group D_4 of order 8.) Let χ be the irreducible character of H of degree 2. The only subgroup of G which properly contains H is G itself, and χ does not extend to G. The character χ^G, of degree 6, is the sum of the two inequivalent characters of G of degree 3.

When H is normal in G, however, the converse does hold.

THEOREM 3: Let H be a normal subgroup of G. Let χ be an irreducible character of H. If χ cannot be extended to any subgroup of G which properly contains H, then χ^G is irreducible.

In this connection, we point out a recent result of Roth [4, Theorem 3.1].

(Roth's) Theorem. Let ξ be a character of G of degree 1. Let $H = ker\ \xi = \{g \epsilon G : \xi(g) = 1\}$. Suppose there exists an irreducible character λ of G such that $\lambda\xi = \lambda$. Then there exists an irreducible character χ on H such that $\chi^G = \lambda$.

IV. Proofs

We begin with Theorem 1. The equivalence of a and b is immediate from the Frobenius Reciprocity Theorem, i.e., $\chi^G = \lambda$ if and only if $(\lambda, \chi)_H = 1$, and $(\xi, \chi)_H = 0$, for every irreducible character ξ of G different from λ. The equivalence of b and c is Theorem (45.2)′ of [1].

The proof of Theorem 2 is equally straight forward. If χ^G is irreducible, then (by Theorem 1) for all $g \epsilon G \setminus H$, χ^g and χ are different on

$$H \cap gHg^{-1} \subset < H, g >,$$

the group generated by H and g. Thus, since characters are class functions, χ cannot be extended to $< H, g >$.

We proceed to the proof of Theorem 3.

LEMMA: Let H be a normal subgroup of G. Let χ be an irreducible character on H. Then χ can be extended to $< H, g >$ if and only if $\chi^g = \chi$.

PROOF: As above, necessity is clear. Suppose, then, that $\chi(g^{-1}hg)=\chi(h)$ for all $h \in H$. Let $h \to A(h)$ be an irreducible representation of H affording χ. Define

$$B(h) = A(g^{-1}hg), \qquad h \in H.$$

Then $h \to B(h)$ is a representation of H which affords χ. It follows that A and B are equivalent. Let U be nonsingular such that

$$B(h) = U^{-1}A(h)U, \qquad h \in H. \tag{1}$$

Now, let r be minimal such that $g^r \in H$. Observe

$$\begin{aligned} A(g^{-r})A(h)A(g^r) &= A(g^{-r}hg^r) \\ &= A(g^{-1}g^{-r+1}hg^{r-1}g) \\ &= B(g^{-r+1}hg^{r-1}) \\ &= U^{-1}A(g^{-r+1}hg^{r-1})U \\ &= \ldots \\ &= U^{-r}A(h)U^r. \end{aligned}$$

Thus, $A(g^r)U^{-r}$ commutes with $A(h)$ for all $h \in H$. It follows from Schur's Lemma that $A(g^r)U^{-r}$ is a scalar matrix S. We now replace U in (1) with U times any scalar rth root of S^{-1}, i.e., we may assume that $U^r=A(g^r)$.

Next, we define R on $<H, g>$ by

$$R(hg^k) = A(h)U^k$$

for all $h \in H$ and $k=0, 1, \ldots, r-1$. We claim R is a representation of $<H, g>$. Observe

$$R(h_1g^s)R(h_2g^t) = A(h_1)U^sA(h_2)U^t \tag{2}$$

and

$$R(h_1g^sh_2g^t) = R(h_1h_2'g^{s+t}) = A(h_1)A(h_2')U^{s+t}, \tag{3}$$

where $h_2'=g^sh_2g^{-s}$. To obtain equality between (2) and (3), it remains to show that

$$U^sA(h_2) = A(h_2')U^s.$$

But, this follows as above. This establishes our claim that R is a representation of $<H, g>$. Since the restriction of R to H is A, the character afforded by R extends χ. The proof of the lemma is complete.

Now, to complete the proof of Theorem 3, we appeal to the implication $c \to b$ of Theorem 1.[3] Since $gHg^{-1}=H$, this implication establishes that χ^G is irreducible if $\chi \neq \chi^g$ for all $g \in G \setminus H$, i.e., by the lemma, if χ cannot be extended to $<H,h>$ for all $g \in G \setminus H$.

COROLLARY: Let H be a normal subgroup of G. Suppose $[G:H]$ is prime. Let λ be an irreducible character of G. Then either the restriction of λ to H is irreducible or $\lambda=\chi^G$ for some irreducible character χ of H.

[3] For a slightly different proof, one could appeal at this point to [2, (9.11)].

5. References

[1] Curtis, Charles, and Reiner, Irving, Representation Theory of Finite Groups And Associative Algebras, (Interscience, New York–London, 1962).
[2] Feit, Walter, Characters of Finite Groups, (Benjamin, New York, 1967).
[3] Merris, Russell, The irreducibility of $K(A)$, Linear and Multilinear Algebra, to appear.
[4] Roth, Richard, On the conjugating representation of a finite group, Pacific J. Math. **36**, 515–521 (1971).

(Paper 78B1–396)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 78B, No. 1, January–March 1974

Integer Arithmetic Determination of Polynomial Real Roots

George W. Reitwiesner

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(May 3, 1973)

The real roots of a polynomial with rational coefficients may be evaluated to absolute precision by integer arithmetic. Based upon the theorems of Sturm and Budan, two algorithms for this evaluation are described, and some comparative observations are offered.

Key words: Budan theorem; exact computation; integer arithmetic; modular arithmetic; polynomial; polynomial real roots; roots; Sturm theorem.

1. Introduction

Given some $P_0(x)$, we consider the sequence $P_0(x)$, $P_1(x)$, . . . $P_t(x)$ of polynomials

$$P_i(x) = \sum p_m^{(i)} x^{n_i - m}$$

formed by an algorithm (detailed below) under which the degrees $n_0, n_1, \ldots n_t$ of the $P_i(x)$ decrease monotonically. We denote degree drop by $d_i = n_i - n_{i-1} \geqq 1$; we distinguish the leading and maximum coefficient magnitudes of the $P_i(x)$ by $c_i = |p_0^{(i)}|$ and $g_i \geqq |p_m^{(i)}|$; we denote the norms of the $P_i(x)$ by $e_i = (\Sigma (p_m^{(i)})^2)^{1/2}$; and we distinguish the degree of the initial $P_0(x)$ by $N = n_0$. We observe $t \leqq N$, and we define $M = N - 1$.

We consider two algorithms. In the Sturm case: $P_1(x)$ is the derivative of $P_0(x)$, and each other $P_i(x)$ is the negative of the remainder polynomial upon dividing $P_{i-2}(x)$ by $P_{i-1}(x)$; and $d_i \geqq 1$ yields $t \leqq N$. In the Budan (or Fourier-Budan) case: every $P_i(x)$ for $i \geqq 1$ is the derivative of $P_{i-1}(x)$; and all $d_i = 1$ yields $t = N$. We discuss these cases, first together, and then separately.

We seek the real roots of $P_0(x)$ by determining the $p_m^{(i)}$ for $i \geqq 1$ and then employing a procedure which requires determining, for selected values of x, the sign(s) of either $P_0(x)$ or all $P_i(x)$. Hence, as convenient, we may replace any $P_i(x)$ by any associate polynomial formed by multiplying $P_i(x)$ by some positive scaling factor f_i. Arbitrarily, we regard $p_0^{(0)} > 0$; and (optionally) we ignore zero roots of $P_0(x)$ by imposing $p_N^{(0)} \neq 0$ (adjusting N if necessary). Trivially, we demand $N > 0$.

By B we denote some bound such that all real roots of $P_0(x)$ lie in the range $-B < x \leqq B$ (e.g., $e_0 < B$); and, by analyzing the signs of the $P_i(x)$ at interval endpoints, we develop successively smaller intervals $x' < x \leqq x''$ within $-B < x \leqq B$ to bound each root in an interval of desired small length.

We consider only rational $p_m^{(0)}$ (e.g., terminating digital expressions), and we restrict all f_i to be rational, whence all $p_m^{(i)}$ are rational.

For exactness in computation, through suitable f_i we constrain all $p_m^{(i)}$ to be integers; and we consider only integer values for B and all x' and x''; hence only integer values are assumed by the $P_i(x)$ at interval endpoints, and the maximum error within which roots ultimately are determinable is 1. Root determination to subinteger precision is tractable, under this integer convention, by suitable initial upscaling of the $p_m^{(i)}$ and corresponding terminal downscaling of the (integer) roots thus determined. Accordingly, hereinafter the minimum value of $x''-x'$ is 1.

Toward precluding unnecessarily large magnitudes among the integer values of the $p_m^{(i)}$ and $P_i(x)$, we remove all common prime factors from the $p_m^{(i)}$ of each $P_i(x)$: initially from $P_0(x)$, and from each other $P_i(x)$ as the algorithm is executed.

The restriction that all values of $p_m^{(i)}$ and B and x' and x'' (and $P_i(x)$) be integers affords opportunity to isolate the roots of $P_0(x)$ into intervals which are bounded exactly—without error accural from numerical rounding or truncation. We are not concerned here with perturbations in root values resulting from change(s) in the integer value(s) of any individual $p_m^{(0)}$.

By $V(x)$ we denote the number of sign changes (ignoring any $P_i(x)=0$) in the sequence $P_0(x)$, $P_1(x)$, . . . $P_t(x)$; and by $R(x', x'')$ we denote the number of real roots of $P_0(x)$ in the interval $x'<x\leqq x''$. Also, we define $\bar{R}(x', x'')=V(x')-V(x'')$. The distinct considerations which apply in establishing $R(x', x'')\leqq\bar{R}(x', x'')$ are discussed separately for the Sturm and Budan cases below.

In both cases, establishing B as any (e.g., the smallest) integer power of 2 which maximizes $\bar{R}(-B,+B)\leqq\bar{R}(-\infty, +\infty)$, and employing $V(x)$ at interval endpoints, the real roots of $P_0(x)$ may be separated into intervals of lengths diminishing to 1—and thus evaluated to within an error of at most 1—by repeated interval halving: first of the initial interval $-B<x\leqq B$, and then of its repeatedly halved subintervals $x'<x\leqq x''$ for which $\bar{R}(x', x'')\neq 0$.

2. The Sturm Case

The Sturm algorithm is expressible as $P_i(x)=[Q_i(x)P_{i-1}(x)-\theta_iP_{i-2}(x)]/\phi_i$ for $i=2, 3, \ldots t$, terminating when $P_{t+1}(x)=0$, where θ_i and ϕ_i are positive scaling factors and the $Q_i(x)$ are discarded.

Obviously $d_1=1$, but $d_i\geqq 1$ applies for $i\geqq 2$; and $t\leqq N$. The constrain that all $p_m^{(i)}$ be integers commends the establishment of $\theta_i=(c_{i-1})^{d_{i-2}+1}$, and $\phi_i>0$ may be chosen arbitrarily so long as the integer quality of the $p_m^{(i)}$ is maintained. Under Euclid's algorithm (for polynomials), the final $P_t(x)$ is (within scaling factors) the greatest-degree polynomial divisor of every $P_i(x)$, and each of its roots is a root of every $P_i(x)$; and, by a property of the derivative, each single or multiple root of $P_0(x)$ is, to multiplicity one less, a root of each $P_1(x)$, $P_2(x)$, . . . $P_t(x)$.

In the Sturm case, every single or multiple root of $P_0(x)$ counting exactly once, $R(x', x'')=\bar{R}(x', x'')=V(x')-V(x'')$ is the number of distinct real roots of $P_0(x)$ in the interval $x'<x\leqq x''$, provided that neither x' nor x'' is a root of $P_t(x)$ (for then would follow all $P_i(x)=0$); however this constraint upon x' and x'' vanishes when every $P_i(x)$ is replaced by its quotient upon division by $(p_0^{(i)}/c_t)P_t(x)$, for then $P_0(x)$ contains no multiple roots, and $P_t(x)=\pm 1$ (and N is correspondingly reduced by n_t).

For the choice all $\phi_i=1$, the $p_m^{(i)}$ may assume unreasonable magnitudes when not all $c_i=1$. The ϕ_i may be chosen [1][1] such that the magnitude of each $p_m^{(i)}$ is equal to that of a particular subdeterminant, of order $2(N-n_i)-1$, of the Sylvester matrix for $P_0(x)$ and $P_1(x)$. This choice is $\phi_i=s_1s_2\ldots s_{i-2Q-2}$, where $h_k=(d_1)(1-d_2)(1-d_3)\ldots(1-d_{k-2})(1-d_{k-1})(d_k)$ and $s_k=(c_k)^{h_k}$; and $\phi_2=1$ is understood; and when all $d_i=1$, there applies simply $\phi_i=(c_{i-2})^2$ for $i>2$. In each of these determinants: only coefficients of $P_1(x)$ occur in $N-n_i$ columns, and only coefficients of $P_0(x)$ occur in $M-n_i$ columns; hence by Hadamard's theorem, each subdeterminant magnitude is bounded by $e_1^{(N-n_i)}e_0^{(M-n_i)}\leqq e_1^Ne_0^M$.

---

[1] Figures in brackets indicate the literature references at the end of this paper.

In implementing the Sturm case, with due sign consideration each $p_m^{(i)}$ may be computed as $\pm$ its counterpart subdeterminant of that Sylvester matrix, and the bound $e_1^N e_0^M$ is excessive at least to the extent that common integer factors occur among the $p_m^{(i)}$ of the $P_i(x)$. Since only $p_0^{(0)}$ and $p_0^{(1)}$ occur in one row of each Sylvester matrix subdeterminant concerned, their greatest common factor $[c_0, c_1]$ is a common factor of all $p_m^{(i)}$ for $i \geqq 2$, and additional common factors may appear in the several $P_i(x)$ as the algorithm is executed. However, in the absence of foreknowledge of such additional factors, the a priori minimum bound of the $p_m^{(i)}$ magnitudes for $i \geqq 2$ is given by $e_1^{(N-n_i)} e_0^{(M-n_i)}/[c_0,c_1] \leqq e_1^N e_0^M/[c_0,c_1]$.

We denote $G = e_1^N e_0^M/[c_0,c_1]$.

Indeed, in the Sturm case, the $p_m^{(i)}$ may be computed [2, 3] as $\pm$ determinants of elements $p_m^{(i-1)}$ and $p_m^{(i-2)}$; and when they are so computed, the magnitudes of intermediately formed products of $p_m^{(i-1)}$ and $p_m^{(i-2)}$ may exceed G, particularly for the larger values of i. However, the bound G does apply for the $p_m^{(i)}$ magnitudes throughout $2 \leqq i \leqq t$, and the generation of the Sturm polynomial coefficients may be conducted to digital precision which need but modestly exceed two times (for sign) the bound G: e.g., modulo some prime which exceeds $2G$; or, to preclude engaging cumbersomely large integers, by a modular arithmetic in which each quantity is represented by its residues, modulo a set of distinct primes whose product exceeds $2G$.

3. The Budan Case

The Budan algorithm [5] is $P_i(x) = d(P_{i-1}(x))/dx$ for $i = 1, 2, \ldots t = N$.

All $d_i = 1$; and, indeed, $P_t(x)$ need not be computed, for it is a constant $P_t(x) = (N!)\, p_0^{(0)}$ of sign identical to that of $p_0^{(0)}$. Clearly, all $p_m^{(i)}$ magnitudes for $i < t$ are bounded by $|p_m^{(i)}| \leqq (M!) g_0 < (M!)\, e_0$.

In the Budan case, every m-fold root of $P_0(x)$ being counted m times, $\bar{R}(x', x'') = V(x') - V(x'')$ is an upper bound on the number of real roots of $P_0(x)$ in the interval $x' < x \leqq x''$, and $R(x', x'')$ can differ from $\bar{R}(x', x'')$ only by an even integer. Therefore $R(x', x'')$ is known precisely only when $R(x', x'') = \bar{R}(x', x'') \leqq 1$; otherwise $R(x', x'') \leqq \bar{R}(x', x'')$ applies.

The potential even integer excess of $\bar{R}(x', x'')$ over $\bar{R}(x', x'')$ accrues from potential complex-conjugate pairs of roots of $P_0(x)$; and the engagement of $\bar{R}(x', x'')$ in isolating the real roots of $P_0(x)$ introduces potential ambiguity, when $\bar{R}(x', x'') \geqq 2$, in distinguishing the occurrence of complex roots of $P_0(x)$ from two cases of the occurrence of real roots in $x' < x \leqq x''$: the case of m-fold real roots, for $m \geqq 2$, which are not expressible as integers under whatever upscaling is employed; and the case of distinct (single or multiple) real roots which lie in the same interval $x' < x \leqq x''$ and differ in value by less than the resolution employed (utlimately $x'' - x' = 1$).

In particular cases, however, such ambiguity may be removable, at least partially, by two devices.

Under Descartes' rule, $P_0(x)$ can have no more positive real roots than the number of sign changes in the sequence $p_0^{(0)}, p_1^{(0)}, \ldots p_N^{(0)}$ and no more negative real roots than in that sequence with every alternate $p_m^{(0)}$ negated.

Also, denoting by $V_m(x)$ the number of sign changes in the sequence $P_t(x), P_{t-1}(x), \ldots P_m(x)$, and defining $\bar{R}_m(x', x'') = V_m(x') - V_m(x'')$, and constraining $m \leqq t-2$: if $P_k(x') > 0$ and $P_k(x'') > 0$ for all $k = t, t-1, \ldots m$, except only $0 > P_{m+1}(x')$, then $\bar{R}_m(x', x'') = 2$, but no real root of $P_m(x)$ can exist in $x' < x \leqq x''$ when the respective magnitudes of $P_m(x')$; and $P_m(x'')$ exceed $x'' - x'$ times those of their tangent slopes $P_{m+1}(x')$ and $P_{m+1}(x'')$ and a corresponding argument applies for <vice>. [And $x'' - x' = 1$ in the ultimate resolution.] The occurrence of every such condition, for a distinct interval $x' < x \leqq x''$, reveals the existence of one pair of complex-conjugate roots of $P_m(x)$, and therefore of $P_0(x)$, for the real roots of every $P_i(x)$ separate those of $P_{i-1}(x)$.

4. Summary

Comparison and/or contrast between the Sturm and Budan algorithms for polynomial real root evaluation is difficult to express, for performance under either is conditioned by the particular set of $p_m^{(0)}$ which comprise $P_0(x)$.

For equal expenditure of digit capacity, in a computer, to accommodate the $p_m^{(i)}$ in either case, and for large N and large (multi-decimal-digit) $p_m^{(0)}$ magnitudes, the bounds $e_1^N e_0^M/[c_0, c_1]$ and $(M!)e_0$ suggest the achievement of higher resolution of real roots under the Budan case, but at risk of uncertainty in the solution whenever $\bar{R}(x', x'') > 1$ occurs for some $x' < x \leqq x''$ at the termination $(x''-x'=1)$ of interval halving.

Under the Sturm case, the precise value of $R(x', x'')$ is known for every interval $x' < x \leqq x''$, but to potentially lower resolution because of the evidently higher a priori bound on the magnitudes of the $p_m^{(i)}$.

In the appendix is displayed a table, summarizing admittedly limited experience for the Sturm case, showing common logarithms of: (a) the bound $e_1^N e_0^M$; (b) the maximum g_i generated in executing the Sturm algorithm for $\phi_i = s_1 s_2 \ldots s_{i-2} c_{i-2}$ but otherwise without common factor removal (without reducing the $p_m^{(i)}$ magnitudes by $[c_0, c_1]$); and (c) the maximum g_i after common factor removal among the coefficients of each $P_i(x)$ [and after division of each $P_i(x)$ by $(p_0^{(i)}/c_i)P_i(x)$]. The first 14 of the 16 entries are for the Nth degree polynomials whose roots are $-1, -2, \ldots -N$ for $N \leqq 14$; the 15th entry is for the 10th degree polynomial with the six real roots $-1/10, -1/11$, 0, 8101/8111, 1, 2, and the two doubled imaginary roots $\pm i$; and the last entry [4] is for the 8th degree polynomial $x^8 - 134x^7 + 6496x^6 - 147854x^5 + 1709659x^4 - 10035116x^3 + 28014804x^2 - 29758896x + 6531840$ whose six real roots are 58.1801 . . ., 32.1652 . . ., 17.6740 . . ., 13.0545 . . ., 1.7103 . . ., 0.2915 . . . . Also shown in the table is the smallest integer j which satisfies $e_0 < B = 2^j$.

For the first 14 entries, the sharpness of the reduction from (a) to (b) and (c) probably is due to the algebraic regularity with which the $p_m^{(0)}$ are expressible—as sums of products of integers (root negatives). For these 14 entries the maximum g_i occurs among the lower values of i, and, through symmetry in this "canonical" case of uniformly spaced roots, each $P_N(x)$ has the final (common factor removed) form $2x + (N+1)$ for even N and $x + (N+1)/2$ for odd N.

For the last two entries, the reduction from (a) to (b) and (c) is less pronounced, probably because of less algebraic regularity among the $p_m^{(0)}$.

However, despite the magnitude of the reduction from (a) to (b) and (c) for the last two entries—no further data [2] are available at this writing—there exists considerable contrast between the magnitudes of (c) (for the Sturm case) and the corresponding common logarithms of the bound $(M!)g_0$ (which would apply for the Budan case): 49.73 versus 11.52 for the 15th entry and 34.93 versus 11.18 for the 16th entry.

This contrast suggests serious consideration of the Budan algorithm, as well as the (infallible) Sturm algorithm, as a mechanism for determining, to absolute precision by integer arithmetic, the real roots of polynomials of large degree with large rational coefficient magnitudes.

5. Appendix

N	(a)	(b)	(c)	j
1	0.0	0.0	0.0	1
2	1.81	.48	.48	2
3	5.98	1.08	1.08	4
4	12.13	1.70	1.70	7
5	24.19	4.92	2.65	9
6	38.93	9.08	3.71	12
7	57.81	14.79	4.42	15
8	76.26	17.38	5.07	18
9	104.65	27.11	6.07	21
10	141.60	42.53	7.47	25
11	179.22	55.65	8.50	28
12	218.35	67.24	10.35	32
13	269.97	88.21	11.11	36
14	311.37	107.80	12.66	40
10	128.51	72.88	49.73	24
8	113.79	61.65	34.93	26

6. References

[1] Collins, George E., Subresultants and reduced polynomial remainder sequences, Journal of Association for Computing Machinery **14**, No. 1, 128–142 (Jan. 1967).

[2] Heindel, Lee E., Integer arithmetic algorithms for polynomial real zero determination, Journal of Association for Computing Machinery **18**, No. 4, 533–548 (Oct. 1971).

[3] Householder, Alston S., Principles of Numerical Analysis (McGraw Hill; 1953), p. 98.

[4] Newman, Morris, An identity for the coefficients of certain modular forms, Journal of London Mathematical Society **30**, 488–493 (1955).

[5] Patrick, Merrell L., A highly parallel algorithm for approximating all zeros of a polynomial with only real zeros, Communications of Association for Computing Machinery **15**, No. 11, 952–955 (Nov. 1972).

(Paper 78B1–397)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematical Sciences
Vol. 78B, No. 1, January-March 1974

Publications of the National Bureau of Standards*

Citations with Selected Abstracts

J. Res. Nat. Bur. Stand. (U.S.), **77A** *(Phys. and Chem.)*, No. 6, (Nov.-Dec. 1973), SD Catalog No. C13.22/sec.A:77/6.

Absolute isotopic abundance ratio and atomic weight of a reference sample of rhenium, J. W. Gramlich, T. J. Murphy, E. L. Garner, and W. R. Shields.

Reflection correction for high-accuracy transmittance measurements on filter glasses, K. D. Mielenz and R. Mavrodineanu.

Temperature-pressure phase relationships in niobium pentoxide, J. L. Waring, R. S. Roth, and H. S. Parker.

Nuclear magnetic resonance of ^{113}Cd and ^{199}Hg in Cd-Mg and Cd-Hg solid solutions, V. V. Zhukow, I. D. Weisman, and L. H. Bennett.

Molecular basis of flame inhibition, J. W. Hastie.

The effect of temperature and pressure on the refractive index of some oxide glasses, R. M. Waxler and G. W. Cleek.

The pair correlation function in liquid ^{4}He, R. D. Mountain and H. J. Raveché.

J. Res. Nat. Bur. Stand. (U.S.), **77B** *(Math. Sci.)*, Nos. 3 and 4, SD Catalog No. C13.22/sec.B:77/3&4.

Certification of an algorithm for Bessel functions of complex argument, D. J. Sookne.

Character induced subgroups, R. Merris and W. Watkins.

The characterizations of $(A_6(U))$, M. E. Sheingorn.

An application of Schur's lemma on irreducible sets of matrices in continuum mechanics, J. T. Fong.

Remarks on a problem of Rademacher in the theory of modular forms, M. I. Knopp.

Performance testing of a FORTRAN library of mathematical function routines—A case study in the application of testing techniques, D. W. Lozier, L. C. Maximon, and W. L. Sadowski.

Bessel functions I and J of complex argument and integer order, D. J. Sookne.

Certification of an algorithm for Bessel functions of real argument, D. J. Sookne.

Triangles generated by powers of triplets on the unit circle, C. R. Johnson and M. Newman.

Tables and graphs of the stable probability density functions, D. R. Holt.

Bessel functions of real argument and integer order, D. J. Sookne.

Monogr. 130. **The divergent beam (Kossel) x-ray method and its uses in measuring strain contours in an individual grain of Fe-3 weight percent Si transformer sheet,** H. Yakowitz, Nat. Bur. Stand. (U.S.), Monogr. 130, 80 pages (Aug. 1973) 95 cents, SD Catalog No. C13.44:130.

Monogr. 131. **Thermal conductivity of solids at room temperature and below. A review and compilation of the literature,** G. E. Childs, L. J. Ericks, and R. L. Powell, Nat. Bur. Stand. (U.S.), Monogr. 131, 624 pages (Sept. 1973) $7.80. SD Catalog No. C13.44:131.

Monogr. 132. **A compilation and evaluation of mechanical, thermal, and electrical properties of selected polymers,** R. E. Schramm, A. F. Clark, and R. P. Reed, Nat. Bur. Stand. (U.S.), Monogr. 132, 848 pages (Sept. 1973) $10.25, SD Catalog No. C13.44:132.

Monogr. 134. **Space groups and lattice complexes,** W. Fisher, H. Burzlaff, E. Hellner, and J. D. H. Donnay, Nat. Bur. Stand. (U.S.), Monogr. 134, 184 pages (May 1973) $4.10. SD Catalog No. C13.44:134.

Monogr. 135. **Properties of glasses in some ternary systems containing BaO and SiO_2,** G. W. Cleek and C. L. Babcock, Nat. Bur. Stand. (U.S.), Monogr. 135, 42 pages (Sept. 1973) 70 cents. SD Catalog No. C13.44:135.

Monogr. 136. **Graphical recoupling of angular momenta,** D. R. Lehman and J. S. O'Connell. Nat. Bur. Stand. (U.S.), Monogr. 136, 18 pages (Oct. 1973) 50 cents. SD Catalog No. C13.44:136.

Monogr. 137. **Applications of waveguide and circuit theory to the development of accurate microwave measurement methods and standards,** R. W. Beatty, Nat. Bur. Stand. (U.S.), Monogr. 137, 322 pages (Aug. 1973) $5.20. SD Catalog No. C13.44:137.

H112. **Examination procedure outlines for commercial weighing and measuring devices. A manual for weights and measures officials,** O. K. Warnlof, Nat. Bur. Stand. (U.S.), Handb. 112, 89 pages (June 1973) $1.10. SD Catalog No. C13.11:112.

SP260-32. **Standard reference materials: Standard quartz cuvettes for high accuracy spectrophotometry,** R. Mavrodineanu and J. W. Lazar, Nat. Bur. Stand. (U.S.), Spec. Publ. 260-32, 27 pages (Dec. 1973). SD Catalog No. C13.10:260-32. In press.

SP300. Volume 10. **Precision measurement and calibration.** Selected NBS papers on image optics, C. S. McCamy, Editor. Nat. Bur. Stand. (U.S.), Spec. Publ. 300. Vol. 10. (Nov. 1972). SD Catalog No. C13.10:300. Vol. 10. In press.

SP305. Supplement 4. **Publications of the National Bureau of Standards 1972 catalog. A compilation of abstracts and key word and author indexes,** B. L. Oberholtzer. Nat. Bur. Stand. (U.S.), Spec. Publ. 305 Suppl. 4, 449 pages (July 1973) $3.75. SD Catalog No. C13.10:305 Suppl. 4.

SP320. Supplement 2. **Bibliography on atomic transition probabilities July 1971 through June 1973,** J. R. Fuhr and W. L. Wiese, Nat. Bur. Stand. (U.S.), Spec. Publ. 320 Suppl. 2, 63 pages (Nov. 1973). 95 cents, SD Catalog No. C13.10:320.

SP374. **Method for determining the resolving power of photographic lenses,** F. E. Washer and I. C. Gardner. Appendix added by C. E. Kuyatt, Nat. Bur. Stand. (U.S.), Spec. Publ. 374, 32 pages (June 1973) $3.00. SD Catalog No. C13.10:374.

SP375. **An index of State specifications and standards, covering those standards and specifications issued by State Purchasing Offices of the United States,** L. L. Grossnickle, Editor. Nat. Bur. Stand. (U.S.), Spec. Publ. 375. 394 pages (Sept. 1973) $3.70. SD Catalog No. C13.10:375.

SP378. **Accuracy in spectrophotometry and luminescence measurements,** *Proc. Conf. held at the National Bureau of Standards, Gaithersburg, Md., Mar. 22-24, 1972,* R. Mavrodineanu, J. I. Shultz, and O. Menis, Editors. Nat. Bur. Stand. (U.S.), Spec. Publ. 378, 268 pages (May 1973) $4.85. SD Catalog No. C13.10:378.

SP381. **Bibliography of ion-molecule reaction rate data (January 1950-October 1971),** G. A. Sinnott, Nat. Bur. Stand. (U.S.), Spec. Publ. 381, 73 pages (Oct. 1973) $1.00. SD Catalog No. C13.10:381.

SP383. **The creative partnership: Government and the professional services,** Proceedings of the Fourth Users-Producers Conference Sponsored by the Technical Analysis Division, Institute for Applied Technology, National Bureau of Standards, in Cooperation with the Center for the Study of Private Enterprise, The American University, held at the National Bureau of Standards, Gaithersburg, Md., January 23, 1973. J. D. Johnson, Ed., Nat. Bur. Stand. (U.S.), Spec. Publ. 383, 142 pages (Aug. 1973) $2.10. SD Catalog No. C13.10:383.

SP384. **Annotated bibliography of the literature on resource sharing computer networks,** R. P. Blanc, I. W. Cotton, T. N. Pyke, Jr., and S. W. Watkins. Nat. Bur. Stand. (U.S.), Spec. Publ. 384, 95 pages (Sept. 1973) $1.25. SD Catalog No. C13.10:384.

SP387. **Laser induced damage in optical materials: 1973.** Proceedings of a symposium sponsored by Office of Naval Research, The American Society for Testing and Materials and by the National Bureau of Standards. Boulder, Colo. May 15-16, 1973. A. J. Glass and A. H. Guenther, Editors. Nat. Bur. Stand. (U.S.), Spec. Publ. 387, 285 pages (Dec. 1973) $2.65. SD Catalog No. C13.10:387.

SP389. **Some references on metric information with charts on all you need to know about metric and metric conversion factors,** W. R. Tilley, Editor. Nat. Bur. Stand. (U.S.), Spec. Publ. 389, 8 pages (Dec. 1973) 25 cents. SD Catalog No. C13.10:389.

AMS63. **Tables of two-associate-class partially balanced designs,** W. H. Clatworthy, J. M. Cameron, and J. A. Speckman, Nat. Bur. Stand. (U.S.), Appl. Math. Ser. 63, 327 pages (May 1973) $3.45. SD Catalog No. C13.32:63.

NSRDS-NBS43. **Selected specific rates of reactions of transients from water in aqueous solution. 1. Hydrated electron,** M. Anbar, M. Bambenek, and A. B. Ross. Nat. Stand. Ref. Data Ser., Nat. Bur. Stand. (U.S.), 43, 67 pages (May 1973) 90 cents. SD Catalog No. C13.48:43.

NSRDS-NBS45. **Radiation chemistry of nitrous oxide gas. Primary processes, elementary reactions, and yields.** G. R. A. Johnson. Nat. Stand. Ref. Data Ser., Nat. Bur. Stand. (U.S.), 45.27 pages (Dec. 1973) 60 cents. SD Catalog No. C13.48:45.

NSRDS-NBS46. **Reactivity of the hydroxyl radical in aqueous solutions,** L. M. Dorfman, G. E. Adams. Nat. Stand. Ref. Data Ser., Nat. Bur. Stand. (U.S.), 46, 72 pages (June 1973) 90 cents. SD Catalog No. C13.48:46.

NSRDS-NBS50. **Resonances in electron impact on atoms and diatomic molecules,** G. J. Schulz, Nat. Stand. Ref. Data Ser., Nat. Bur. Stand. (U.S.), 50, 118 pages (Oct. 1973) $1.35. SD Catalog No. C13.48:50.

BSS45. **Dynamic thermal performance of an experimental masonry building,** B. A. Peavy, F. J. Powell, and D. M. Burch. Nat. Bur. Stand. (U.S.), Bldg. Sci. Ser. 45, 103 pages (July 1973) $1.25. SD Catalog No. C13.29/2:45.

BSS46. **Building practices for disaster mitigation.** Proceedings of a workshop sponsored by The National Science Foundation, Research Applied to National Needs Program, and The National Bureau of Standards, held at the National Bureau of Standards, Boulder, Colo., Aug. 28-Sept. 1, 1972, R. Wright, S. Kramer, and C. Culver, Editors. Nat. Bur. Stand. (U.S.), Bldg. Sci. Ser. 46, 483 pages (Feb. 1973) $5.30. SD Catalog No. C13.29/2:46.

BSS47. **Structural deflections. A literature and state-of-the-art survey,** T. V. Galambos, P. L. Gould, M. K. Ravindra, H. Suryoutomo, and R. A. Crist, Nat. Bur. Stand. (U.S.), Bldg. Sci. Ser. 47, 104 pages (Oct. 1973) $1.25. SD Catalog No. C13.29/2:47.

FIPS PUB 3-1. **Recorded magnetic tape for information exchange (800 CPI, NRZI),** P. S. Johnson, Standards Coordinator. Nat. Bur. Stand. (U.S.), Fed. Info. Process. Stand. Publ. (FIPS PUB) 3-1, 4 pages (1973) 20 cents. SD Catalog No. C13.52:3-1.

FIPS PUB 6-2. **Counties and county equivalents of the states of the United States. Federal general data standard representations and codes,** *(ANS X3.31-1973),* H. E. McEwen, Standards Coordinator. Nat. Bur. Stand. (U.S.), Fed. Info. Process. Stand. Publ. (FIPS PUB) 6-2, 35 pages (1973) 65 cents. SD Catalog No. C13.52:6-2.

FIPS PUB 8-3. **Standard metropolitan statistical areas,** H. E. McEwen, Standards Coordinator. Nat. Bur. Stand. (U.S.), Fed. Info. Process. Stand. Publ. (FIPS PUB) 8-3. 20 pages (1973) 55 cents. SD Catalog No. C13.52:8-3.

FIPS PUB 25. **Recorded magnetic tape for information interchange (1600 CPI, phase encoded),** P. S. Johnson, Standards Coordinator. Nat. Bur. Stand. (U.S.), Fed. Info. Process. Stand. Publ. (FIPS PUB) 25, 4 pages (1973) 20 cents. SD Catalog No. C13.52:25.

FIPS PUB 26. **One-inch perforated paper tape for information interchange,** P. S. Johnson, Standards Coordinator. Nat. Bur. Stand. (U.S.), Fed. Info. Process. Stand. Publ. (FIPS PUB) 26, 4 pages (1973) 20 cents. SD Catalog No. C13.52:26.

FIPS PUB 27. **Take-up reels for one-inch perforated tape for information interchange,** P. S. Johnson, Standards Coordinator. Nat. Bur. Stand. (U.S.), Fed. Info. Process. Stand. Publ. (FIPS PUB) 27, 4 pages (1973) 20 cents. SD Catalog No. C13.52:27.

PS54-72. **Body measurements for the sizing of girls' apparel,** C. W. Devereux, II, Technical Standards Coordinator. Nat. Bur. Stand. (U.S.), Prod. Stand. 54-72, 19 pages (July 1973) 30 cents. SD Catalog No. C13.20/54-72.

PS56-73. **Structural glued laminated timber,** *(ANS A190. 1-1973),* K. G. Newell, Jr., Technical Standards Coordinator. Nat. Bur. Stand. (U.S.), Prod. Stand. 56-73. 12 pages (Oct. 1973) 45 cents. SD Catalog No. C13.20/2:56-73.

TN392. (Revised September 1973). **Thermodynamic properties of compressed gaseous and liquid fluorine,** R. Prydz and G. C. Straty, Nat. Bur. Stand. (U.S.), Tech. Note 392, 182 pages $1.50, SD Catalog No. C13.46:392 (Rev. 1973).

TN594-5. **Optical Radiation Measurements: Stability and temperature characteristics of some silicon and selenium photodetectors,** K. Mohan, A. R. Schaefer, and E. F. Zalewski. Nat. Bur. Stand. (U.S.), Tech. Note 594-5, 16 pages (June 1973) 35 cents. SD Catalog No. C13.46:594-5.

TN620. **Frequency and phase stabilization of an HCN laser by locking to a synthesized reference,** J. S. Wells and D. Halford.

Nat. Bur. Stand. (U.S.), Tech. Note 620. 56 pages (May 1973) 60 cents. SD Catalog No. C13.46:620.

TN638. **A synchronous satellite time delay computer,** W. F. Hamilton and D. W. Hanson, Nat. Bur. Stand. (U.S.), Tech. Note 638, 39 pages (July 1973) 45 cents. SD Catalog No. C13.46:638.

TN639. **Publications and services of the National Bureau of Standards, Cryogenics Division, Institute for Basic Standards, Boulder, Colo. 80302, 1953-1972,** J. R. Mendenhall. V. J. Johnson, and N. A. Olien, Nat. Bur. Stand. (U.S.), Tech. Note 639. 82 pages (Aug. 1973) 75 cents. SD Catalog No. C13.46:639.

TN640. **Considerations for the precise measurement of amplifier noise,** D. F. Wait, Nat. Bur. Stand. (U.S.), Tech. Note 640. 129 pages (Aug. 1973) $1.25. SD Catalog No. C13.10.46:640.

TN641. **Survey of the properties of the hydrogen isotopes below their critical temperatures,** H. M. Roder. G. E. Childs. R. D. McCarty, and P. E. Angerhofer, Nat. Bur. Stand. (U.S.), Tech. Note 641, 122 pages (Oct. 1973) $1.25. SD Catalog No. C13.46:641.

TN642. **Summary of WR15 flange evaluation at 60 GHz,** B. C. Yates and G. J. Counas. Nat. Bur. Stand. (U.S.), Tech. Note 642, 32 pages (Oct. 1973). SD Catalog No. C13.46:642. In press.

TN643. **Measurement of rf power and attenuation using superconducting quantum interference devices,** R. A. Kamper. M. B. Simmonds. C. A. Hoer. and R. T. Adair. Nat. Bur. Stand. (U.S.), Tech. Note 643. 93 pages (Aug. 1973) $1.00. SD Catalog No. C13.46:643.

TN644. **Application of a non-ideal sliding short to two-port loss measurement,** M. P. Weidman and G. F. Engen, Nat. Bur. Stand. (U.S.), Tech. Note 644, 40 pages (Oct. 1973) 50 cents. SD Catalog No. C13.46:644.

TN645. **Time and frequency broadcast experiments from the ATS-3 satellite,** D. W. Hanson and W. F. Hamilton, Nat. Bur. Stand. (U.S.), Tech. Note 645. 115 pages (Nov. 1973) $1.00. SD Catalog No. C13.46:645.

TN740. **SETAB: An edit insert program for automatic typesetting of spectroscopic and other computerized tables,** R. C. Thompson and J. Hilsenrath. Nat. Bur. Stand. (U.S.), Tech. Note 740. 30 pages (Dec. 1973) 55 cents. SD Catalog No. C13.46:740.

TN752. **Directory of law enforcement and criminal justice associations and research centers,** B. J. Latka. Nat. Bur. Stand. (U.S.), Tech. Note 752, 49 pages (June 1973) 80 cents. SD Catalog No. C13.46:752.

TN759. **The NBS lead paint poisoning project: Housing and other aspects,** H. W. Berger. Nat. Bur. Stand. (U.S.), Tech. Note 759. 25 pages (Feb. 1973) 30 cents. SD Catalog No. C13.46:759.

TN767. **Microelectronic interconnection bonding with ribbon wire,** H. K. Kessler and A. H. Sher. Nat. Bur. Stand. (U.S.), Tech. Note 767. 31 pages (Apr. 1973) 50 cents. SD Catalog No. C13.46:767.

TN769. **A rigorous correction procedure for quantitative electron probe microanalysis (COR 2),** J. Hénoc, K. F. J. Heinrich. and R. L. Myklebust, Nat. Bur. Stand. (U.S.), Tech. Note 769. 132 pages (Aug. 1973) $2.10. SD Catalog No. C13.46:769.

TN773. **Methods of measurement for semiconductor materials, process control, and devices. Quarterly report, October 1 to December 31, 1972,** W. M. Bullis. Editor. Nat. Bur. Stand. (U.S.), Tech. Note 773. 56 pages (June 1973) 80 cents. SD Catalog No. C13.46:773.

TN774. **Fire service location-allocation models,** D. Colner and D. Gilsinn, Nat. Bur. Stand. (U.S.), Tech. Note 774, 46 pages (June 1973) 75 cents. SD Catalog No. C13.46:774.

TN776. **A new method for generating waterdrops of specified mass,** J. E. Potzick. Nat. Bur. Stand. (U.S.), Tech. Note 776, 12 pages (May 1973) 35 cents. SD Catalog No. C13.46:776.

TN777. **Cost analysis of blood banking alternatives,** T. Miller. M. King, J. Flannagan, E. Nilsson, and B. Lemieux. Nat. Bur. Stand. (U.S.), Tech. Note 777. 121 pages (Sept. 1973) $1.35. SD Catalog No. C.13.46:777.

TN779. **Data communications system throughput performance using high speed terminals on the dial telephone network,** D. S. Grubb, Nat. Bur. Stand. (U.S.), Tech. Note 779. 40 pages (May 1973) 65 cents. SD Catalog C13.46:779.

TN780. **Controlled accessibility bibliography,** S. K. Reed and M. M. Gray, Nat. Bur. Stand. (U.S.), Tech. Note 780. 15 pages (June 1973) 35 cents. SD Catalog No. C13.46:780.

TN781. **A study of six university-based information systems,** B. Marron, E. Fong, D. W. Fife, and K. Rankin, Nat. Bur. Stand. (U.S.), Tech. Note 781, 98 pages (June 1973) $1.25. SD Catalog No. C13.46:781.

TN783. **Durability and maintenance as related to the selection of flooring,** W. C. Wolfe, R. F. Roberts. and M. Russell. Nat. Bur. Stand. (U.S.), Tech. Note 783. 68 pages (Aug. 1973) 90 cents. SD Catalog No. C13.46:783.

TN785. **Scanning electron microscope examination of wire bonds from high-reliability devices,** K. O. Leedy, Nat. Bur. Stand. (U.S.), Tech. Note 785. 35 pages (Aug. 1973) 55 cents. SD Catalog No. C13.46:785.

TN786. **Methods for testing wire-bond electrical connections,** H. A. Schafft, Nat. Bur. Stand. (U.S.), Tech. Note 786, 23 pages (Nov. 1973) 50 cents. SD Catalog No. C13.46:786.

TN787. **Heuristic cost optimization of the Federal Telpak network,** R. G. Saltman, G. R. Bolotsky, and Z. G. Ruthberg, Nat. Bur. Stand. (U.S.), Tech. Note 787. 52 pages (June 1973) 80 cents. SD Catalog No. C13.46:787.

TN788. **Methods of measurement for semiconductor materials, process control, and devices, quarterly report January 1 to March 31, 1973,** W. M. Bullis. Editor. Nat. Bur. Stand. (U.S.), Tech. Note 788, 79 pages (Aug. 1973) 95 cents. SD Catalog No. C13.46.788.

TN789. **Technical options for energy conservation in buildings,** National Conference of States on Building Codes and Standards and National Bureau of Standards Joint Emergency Workshop on Energy Conservation in Buildings held at the U.S. Department of Commerce Washington, D.C., June 19. 1973. Nat. Bur. Stand. (U.S.), Tech. Note 789. 184 pages (July 1973) $2.35. SD Catalog No. C13.46:789.

TN790. **MIDAS modular interactive data acquisition system—description and specification,** C. H. Popenoe and M. S. Campbell. Nat. Bur. Stand. (U.S.), Tech. Note 790. 49 pages (Aug. 1973) 75 cents. SD Catalog No. C13.46:790.

TN791. **Dose calibrator pilot study,** S. B. Garfinkel and G. J. Hine, Nat. Bur. Stand. (U.S.), Tech. Note 791, 7 pages (Aug. 1973) 30 cents. SD Catalog No. C13.46:791.

TN792. **Investigation of viscous flow in glass during phase separation,** J. H. Simmons. S. A. Mills. A. Napolitano, D. H. Blackburn, and W. K. Haller. Nat. Bur. Stand. (U.S.), Tech. Note 792, 43 pages (Sept. 1973), 65 cents. SD Catalog No. C13.46:792.

TN793. **Development of a procedure for measuring the noise of paper caps,** M. A. Cadoff, D. E. Mathews. and D. S. Blomquist. Nat. Bur. Stand. (U.S.), Tech. Note 793. 23 pages (Oct. 1973) 50 cents. SD Catalog No. C13.46:793.

TN794. **NBS corridor fire tests: Energy and radiation models,** F. C. W. Fung, R. Suchomel. and P. L. Oglesby, Nat. Bur. Stand. (U.S.), Tech. Note 794, 127 pages (Oct. 1973) $1.40. SD Catalog No. C13.46:794.

TN795. **Review of network management problems and issues,** A. J. Neumann, Nat. Bur. Stand. (U.S.), Tech. Note 795. 77 pages (Oct. 1973) $1.00. SD Catalog No. C13.46:795.

TN796. **FRAME: An on-line correction procedure for quantitative electron probe microanalysis,** H. Yakowitz, R. L. Myklebust, and K. F. J. Heinrich. Nat. Bur. Stand. (U.S.), Tech. Note 796, 51 pages (Oct. 1973) 80 cents. SD Catalog No. C13.46:796.

TN797. **Static language analysis,** G. Lyon, Nat. Bur. Stand. (U.S.), Tech. Note 797. 23 pages (Oct. 1973) 50 cents. SD Catalog No. C13.46:797.

TN798. **Collaborative research program between NBS and Indian Scientific Institutions. Special foreign currency program 1973 status,** H. S. Peiser. M. B. McNeil. and D. M. Bluebond. Editors. Nat. Bur. Stand. (U.S.), Tech. Note 798, 139 pages (Nov. 1973) $1.50. SD Catalog No. C13.46:798.

TN799. **User procedures standardization for network access,** A. J. Neumann, Nat. Bur. Stand. (U.S.), Tech. Note 799. 43 pages (Oct. 1973) 70 cents. SD Catalog No. C13.46:799.

TN802. **Network user information support,** A. J. Neumann, Nat. Bur. Stand. (U.S.), Tech. Note 802, 27 pages (Dec. 1973) 60 cents. SD Catalog No. C13.46:802.

TN806. **Methods of measurement for semiconductor materials, process control, and devices. Quarterly report, April 1 to June 30, 1973,** W. M. Bullis. Editor. Nat. Bur. Stand. (U.S.), Tech. Note 806, 77 pages (Nov. 1973) $1.00. SD Catalog No. C13.46:806.

TN807. **Building performance in the 1972 Managua earthquake,** R. N. Wright and S. Kramer. Nat. Bur. Stand. (U.S.), Tech. Note 807. 155 pages (Nov. 1973) $1.60. SD Catalog No. C13.46:807.

TN808. **Potential systems for lead hazard elimination: Evaluations and recommendations for use,** D. Waksman, J. B. Ferguson, M. Godette, and T. Reichard. Nat. Bur. Stand. (U.S.), Tech. Note 808. 192 pages (Dec. 1973) $1.95. SD Catalog No. C13.46:808.

TN810. **Fire incidents involving sleepwear worn by children ages 6-12,** J. A. Slater. Nat. Bur. Stand. (U.S.), Tech. Note 810. 23 pages (Dec. 1973) 50 cents. SD Catalog No. C13.46:810.

CIS7. **What About Metric?,** L. E. Barbrow. Nat. Bur. Stand. (U.S.), Consum. Inf. Ser. 7. 16 pages (Oct. 1973) 80 cents. SD Catalog No. C13.53:7.

This column lists all outside publications by the NBS staff, as soon after issuance as practical. For completeness, earlier references not previously reported may be included from time to time.

Abrams. M. D., **Remote computing: The administrative side,** *Comput. Decisions*. pp. 42-46 (Oct. 1973).

Achenbach. P. R., Phillips. C. W., **Performance characteristics of pressure-actuated water-regulating valves for refrigerant condensers,** (Proc. XII Int. Congress of Refrigeration, Madrid. Spain, 1967), Paper in *Progress in Refrigeration Science and Technology* **2,** 1107-1118 (Gráficas Reunidas. Madrid. Spain, 1969).

Acquisita, N., Abramowitz, S., **Vibrational spectrum of MoF_5,** *J. Chem. Phys.* **58,** No. 12, 5484-5488 (June 15. 1973).

Albers. J., Deutch. J. M., **On the rate equation description of spectral lines,** *Chem. Phys.* **1,** No. 2, 89-98 (Mar./Apr. 1973).

Anderson, R. L., **The high temperature stability of platinum resistance thermometers,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4,** Part 2. 927-934 (Instrument Society of America, Pittsburgh. Pa., 1972).

Aras. N. K., Zoller. W. H., Gordon, G. E., Lutz, G. J., **Instrumental photon activation analysis of atmospheric particulate material,** *Anal. Chem.* **45,** No. 8, 1481-1489 (July 1973).

Armstrong, G. T., **Calorimetric reference materials—status of the primary standard,** *Proc. 1st Int. Conf. on Calorimetry and Thermodynamics, Warsaw, Poland. Aug. 31-Sept. 4, 1969*. pp. 261-267 (Polish Scientific Publishers. Warsaw, Poland. 1971).

Artru. M-C., Kaufman, V., **Re-evaluation of parameters for even configurations of triply ionized magnesium,** *J. Opt. Soc. Amer.* **63,** No. 4, 449-450 (Apr. 1973).

Arvidson, J. M., Durcholz, R. L., Reed. R. P., **Compressive properties of polyurethane and polystyrene foams from 76 to 300 K,** (Proc. 1972 Cryogenic Engineering Conf., Boulder. Colo., Aug. 9-11, 1972), Paper E-4 in *Advances in Cryogenic Engineering* **18,** 194-201 (Plenum Press. New York. N.Y., 1973).

Atkinson, G. H., Laufer. A. H., Kurylo, M. J., **Detection of free radicals by an intracavity dye laser technique,** *J. Chem. Phys.* **59,** No. 1, 350-354 (July 1, 1973).

Baker. M. A., **Observing porcelain enamels with a scanning electron microscope,** *Proc. Porcelain Enamel Institute Technical Forum, The Ohio State University, Columbus, Ohio, Oct. 6-8, 1971,* **33,** 84-90 (Porcelain Enamel Institute, Inc., Washington, D.C., 1971).

Baker. M. A., **Weatherability of porcelain enamels,** (Proc. Aluminum Finishing Seminar. Chicago, Ill., Mar. 19-21, 1973), Paper C12 in Applied Session: Technical Papers. pp. C12-1—C12-16 (The Aluminum Ass., New York. N.Y., Mar. 1973).

Baker. M. A., **Weathering tests of porcelain enamels on steel and aluminum,** *Proc. Porcelain Enamel Institute Technical Forum, The University of Illinois. Urbana, Ill., Oct. 11-13. 1972,* **34,** 186-198 (Porcelain Enamel Institute, Inc., Washington, D.C., 1972).

Ballantyne, J. P., Yakowitz, H., Nixon, W. C., **Simultaneous x-ray microanalysis and resistance measurement of electron**

beam induced direct metallic deposition, (Proc. 6th Int. Conf. on X-Ray Optics and Microanalysis. Osaka, Japan, Sept. 1971), Paper in *Proceedings of the Sixth International Conference on X-Ray Optics and Microanalysis*. G. Shinoda, K. Kohra and T. Ichinokawa, Eds., pp. 219-227 (University of Tokyo Press. Tokyo, Japan, 1972).

Ballard. L. D., Edelman, S., Epstein, W. S., Smith. E. R., **A suggestion for determining g by a two interferometer technique,** *SPIE J.* **9,** No. 5. 166-168 (June/July 1971).

Barber. W. C., Hayward. E., Sazama, J., **Nuclear scattering of plane-polarized photons,** (Proc. Int. Conf. on Nuclear Structure Studies using Electron Scattering and Photoreaction, Sendai. Japan, Sept. 12-15. 1972), Paper in *Nuclear Structure Studies Using Electron Scattering and Photoreaction*, Supplement to *Research Report of Laboratory of Nuclear Science*, K. Shoda and H. Ui. Eds., **5,** 313-317 (Tohoku University, Tomizawa, Sendai. Japan, 1972).

Barger, R. L., Sorem, M. S., Hall, J. L., **Frequency stabilization of a cw dye laser,** *Appl. Phys. Lett.* **22,** No. 11, 573-575 (June 1, 1973).

Barnes, I. L., Murphy, T. J., Gramlich, J. W., Shields, W. R., **Lead separation by anodic deposition and isotope ratio mass spectrometry of microgram and smaller samples,** *Anal. Chem.* **45,** No. 11, 1881-1884 (Sept. 1973).

Barnes, J. D., **Inelastic neutron scattering study of the "rotator" phase transition in *n*-nonadecane,** *J. Phys. Chem.* **58,** No. 12, 5193-5201 (June 15, 1973).

Barnes, J. D., **Neutron inelastic scattering study of the "rotator" phase transition in n-alkanes,** (Proc. 5th Symp. on Neutron Inelastic Scattering, Grenoble, France, Mar. 6-10. 1972). Chapter in *Neutron Inelastic Scattering*, pp. 287-299 (International Atomic Energy Agency, Vienna, Austria, 1972).

Barton, J. A., Jr., Burns. C. L., Chandler. H. H., Bowen, R. L., **An experimental radiopaque composite material,** *J. Dent. Res.* **52,** No. 4, 731-739 (1973).

Becker, D. A., LaFleur, P. D., **Neutron activation analysis: Application to trace element analysis in biological and environmental materials,** *Proc. 5th Annual Conf. on Trace Substances in Environmental Health, University of Missouri. Columbia, Mo., 1971*, pp. 447-453 (1972).

Bedford, R. E., Ma, C. K., Barber, C. R., Chandler, T. R., Quinn, T. J., Burns, G. W., Scroger, M., **New reference tables for platinum 10% rhodium/platinum and platinum 13% rhodium/platinum thermocouples,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4,** Part 3, 1585-1602 (Instrument Society of America, Pittsburgh, Pa., 1972).

Bellet, J., Lafferty, W. J., Steenbeckeliers, G., **Microwave spectra of $D_2{}^{17}O$ and $D_2{}^{18}O$,** *J. Mol. Spectrosc.* **47,** No. 3, 388-402 (Sept. 1973).

Benzinger, M., Benzinger, T. H., **Tympanic clinical temperature,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4,** Part 3, 2089-2102 (Instrument Society of America, Pittsburgh, Pa., 1972).

Benzinger, T. H., **A new concept in thermodynamics and its implication in molecular biology and pharmacology,** Chapter 14 in *Methods in Pharmacology* **2,** Part 2, 481-488 (Appleton-Century Crofts, New York, N.Y., 1972).

Berg, C. A., **Energy conservation through effective utilization,** *NBSIR 73-102*, 53 pages (Feb. 1973). (Available as COM 73-10856 from the National Technical Information Service, Springfield, Va. 22151.)

Birky, M. M., **Calorimeter for laser energy measurements,** *Appl. Opt.* **10,** No. 1, 132-135 (Jan. 1971).

Blomquist, D. S., **An experimental investigation of foam windscreens,** (Proc. 1973 International Noise Control Engineering Conference, Copenhagen, Denmark. Aug. 22-24, 1973). Paper in *Inter-Noise 73 Proceedings*. O. J. Pedersen, Ed., pp. 589-593 (Inter-Noise 73. Technical University, Lyngby, Denmark. 1973).

Blomquist, D. S., Leasure, W. A., Jr., **An hourly noise exposure meter,** (Proc. 1973 International Noise Control Engineering Conference, Copenhagen, Denmark. Aug. 22-24, 1973). Paper in *Inter-Noise 73 Proceedings*. O. J. Pedersen, Ed., pp. 67-69 (Inter-Noise 73. Technical University, Lyngby, Denmark. 1973).

Boone, T. H., Ray, T. R., Street, W. G., **Pilot demonstration of lead based paint hazard elimination methods,** *NBSIR 73-242*, 38 pages (June 1973). (Available as PB 224654 from the National Technical Information Service, Springfield. Va. 22151.)

Bowen, B. E., Cram. S. P., Leitner. J. E., Wade, R. L., **High precision sampling for chromatographic separations,** *Anal. Chem.* **45,** No. 13. 2185-2191 (Nov. 1973).

Bowen, R. L., Argentar. H., **A method for determining the optimum peroxide-to-amine ratio for self-curing resins,** *J. Appl. Polym. Sci.* **17,** 2213-2222 (1973).

Bowen, R. L., Chandler, H. H., **Metal-filled resin composites,** *J. Dental Res.* **52,** No. 3. 522-532 (May-June 1973).

Brauer, G. M., Termini, D. J., **Grafting of acrylates and vinyl chains onto collagen with ceric initiator,** *J. Appl. Polym. Sci.* **17,** 2557-2568 (1973).

Brenner, F. C., Kondo, A., Cohen, G. B., **Research for a uniform quality grading system for tires V. Effect of environment on tread wear rate,** *Rubber Chem. Technol.* **44,** No. 4, 952-959 (Sept. 1971).

Broadhurst, M. G., **Fluctuation-barrier model for rotational relaxation,** (Proc. Conf. Electrical Insulation and Dielectric Phenomena, Buck Hill Falls, Pa., Oct. 20-22, 1969), Chapter in *1969 Annual Report of the Conference Electrical Insulation and Dielectric Phenomena*, pp. 48-54 (National Research Council, National Academy of Sciences, Washington, D.C., 1970).

Broadhurst, M. G., Malmberg, C. G., Mopsik, F. I., Harris, W. P., **Piezo- and pyroelectricity in polymer electrets,** (Proc. Conf. on Electrets, Charge Storage and Transport in Dielectrics. Miami Beach, Fla., Oct. 8-13, 1972), Paper in *Electrets, Charge Storage and Transport in Dielectrics*, M. M. Perlman, Ed., pp. 492-504 (The Electrochemical Society, Inc., Princeton, N.J., 1973). (Available as AD 758730 from the National Technical Information Service, Springfield. Va. 22151).

Broadhurst, M. G., Mopsik, F. I., **Vibrational frequency spectrum for polymers,** *J. Chem. Phys.* **55,** No. 8, 3708-3711 (Oct. 15. 1971).

Brower, W. S., Parker, H. S., Roth. R. S., **Synthesis of mercury bismuth sulfide $HgBi_2S_4$,** *Mater. Res. Bull.* **8,** 859-862 (1973).

526-390 O - 74 - 4

Brown, W. C., Buchanan, C. J., **A study of the strength capabilities of children ages two through six,** *NBSIR 73-156*, 56 pages. (Aug. 7, 1973). (Available as COM73-11286 from the National Technical Information Service, Springfield, Va. 22151.)

Brown, W. E., **Solubilities of phosphates and other sparingly soluble compounds,** Chapter 10 in *Environmental Phosphorus Handbook*, E. J. Griffith, A. Beeton, J. M. Spencer and D. T. Mitchell, Eds., pp. 203-239 (John Wiley & Sons, Inc., New York, N.Y., 1973).

Bur, A. J., **Dielectric properties of fluorine-containing polymers,** Chapter 15 in *Fluoropolymers*, L. A. Wall, Ed., **25**, 475-505 (John Wiley-Interscience, New York, N.Y., 1972).

Burns, G. W., Hurst, W. S., **Studies of the performance of W-RE type thermocouples,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4**, Part 3, 1751-1766 (Instrument Society of America, Pittsburgh, Pa., 1972).

Bussey, H. E., **Wavelength of a slotted rectangular line containing two dielectrics,** *NBSIR 73-326*, 17 pages (July 1973). (Available as COM 73-11465-AS from the National Technical Information Service, Springfield, Va. 22151.)

Candela, G. A., Kahn, A. H., Negas, T., **Magnetic susceptibility of $Co^{++}(d^5)$ in octahedral and tetrahedral environments,** *J. Solid State Chem.* **7**, No. 4, 360-369 (1973).

Canfield, L. R., Johnston, R. G., Madden, R. P., **NBS detector standards for the far ultraviolet,** *Appl. Opt.* **12**, No. 7, 1611-1617 (July 1973).

Carpenter, B. S., Samuel, D., Wasserman, I., **Quantitative applications of ^{17}O tracer,** *Radiat. Eff. Short Commun.* **19**, 59 (1973).

Cataland, G., Plumb, H. H., **The realization of low temperature fixed points on the NBS acoustical and the NBS-1955 temperature scales,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4**, Part 1, 183-193 (Instrument Society of America, Pittsburgh, Pa., 1972).

Cattaneo, L. E., Yokel, F. Y., **Structural tests of mechanical connections for concrete panels,** *NBSIR 73-126*, 118 pages (Nov. 1972). (Available as PB 219124 from the National Technical Information Service, Springfield, Va. 22151.)

Cezairliyan, A., **A high-speed (millisecond) system for studies of phase transitions and thermophysical properties of electrical conductors above 2000 K,** *Proc. Int. Conf. on Etude des Transformations Cristallines a Haute Température, Odeillo, France, Sept. 27-30, 1971*, No. 205, 25-32 (Centre National de la Recherche Scientifique, Paris, France, 1972).

Cezairliyan, A., **High-speed (subsecond) simultaneous measurement of specific heat, electrical resistivity, and hemispherical total emittance of Ta-10 (wt.%)W alloy in the range 1500 to 3200 K,** *High Temp.-High Pressures* **4**, 541-550 (1972).

Cezairliyan, A., **Measurement of the heat capacity of graphite in the range 1500 to 3000 K by a pulse heating method,** (Proc. 6th Symp. on Thermophysical Properties, Palm Springs, Calif., Aug. 1973), Paper in *Proceedings of the 6th Symposium on Thermophysical Properties*, P. E. Liley, Ed., pp. 279-285 (American Society of Mechanical Engineers, New York, N.Y., 1973).

Cezairliyan, A., **Measuring transient high temperatures by optical pyrometry,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4**, Part 1, 657-664 (Instrument Society of America, Pittsburgh, Pa., 1972).

Chertok, B. T., Sheffield, C., Lightbody, J. W., Jr., Penner, S., Blum, D., **Low-q^2 electron scattering from the 15.109-MeV state of ^{12}C and the conserved-vector-current test,** *Phys. Rev. C* **8**, No. 1, 23-36 (July 1973).

Chesler, S. N., Cram, S. P., **Iterative curve fitting of chromatographic peaks,** *Anal. Chem.* **45**, No. 8, 1354-1359 (July 1973).

Chignell, C. F., Benzinger, T. H., **Heatburst microcalorimetry,** Chapter 14 in *Methods in Pharmacology* **2**, Part 1, 465-489 (Appleton-Century Crofts, New York, N.Y., 1972).

Christ, B. W., **Effects of misalignment on the pre-macroyield region of the uniaxial stress-strain curve,** *Met. Trans.* **4**, 1961-1965 (Aug. 1973).

Churney, K. L., West, E. D., Armstrong, G. T., **A cell model for isoperibol calorimeters,** *NBSIR 73-184*, 100 pages (Apr. 1973). (Available as COM 73-11110 from the National Technical Information Service, Springfield, Va. 22151.)

Cohen, J., Edelman, S., Vezzetti, C. F., **Pyroelectricity and piezoelectricity in oriented films of polyvinyl fluoride and polyvinylidene fluoride,** (Proc. Conf. on Electrets, Charge Storage and Transport in Dielectrics, Miami Beach, Fla., Oct. 8-13, 1972), Paper in *Electrets, Charge Storage and Transport in Dielectrics*, Martin M. Perlman, Ed., pp. 505-516 (The Electrochemical Society, Inc., Princeton, N.J., 1973).

Cohen, E. R., Taylor, B. N., **A reevaluation of the fundamental physical constants,** (Proc. 4th Int. Conference on Atomic Masses and Fundamental Constants, Teddington, England, Sept. 3-7, 1971), Paper in *Atomic Masses and Fundamental Constants*, J. H. Sanders and A. H. Wapstra, Eds., Part 13, 543-563 (Plenum Publishing Co., New York, N.Y., 1972).

Collier, R. S., Ellerbruch, D., Cruz, J. E., Stokes, R. W., Luft, P. E., Peterson, R. G., Hiester, A. E., **Mass quantity gauging by rf mode analysis,** *NBSIR 73-318*, 196 pages (June 1973). (Available as N73-27390 from the National Technical Information Service, Springfield, Va. 22151.)

Collins, R. C., Haller, W., **Protein-sodium dodecyl sulfate complexes: Determination of molecular weight, size and shape by controlled pore glass chromatography,** *Anal. Biochem.* **54**, 47-53 (1973).

Comeford, J. J., Birky, M., **A method for the measurement of smoke and HCl evolution from poly(vinyl chloride),** *Fire Technol.* **8**, No. 2, 85-90 (May 1972).

Coriell, S. R., Sekerka, R. F., **Morphological stability near a grain boundary groove in a solid-liquid interface during solidification of a binary alloy,** *J. Cryst. Growth* **19**, 285-293 (1973).

Coriell, S. R., Sekerka, R. F., **Morphological stability near a grain boundary groove in a solid-liquid interface during solidification of a pure substance,** *J. Cryst. Growth* **19**, 90-104 (1972).

Corliss, E. L. R., **Remark on "Fixed-scale mechanism of absolute pitch",** *J. Acoust. Soc. Amer. Letters to Editor* **53**, No. 6, 1737-1739 (June 1973).

Costrell, L., Editor. **CAMAC—A modular instrumentation system for data handling,** (AEC Committee on Nuclear Instrument Modules), *AEC Report No. TID-25875*, 66 pages (U.S. Atomic Energy Commission, Washington, D.C., July 1972).

Costrell, L., Editor. **Supplementary information on CAMAC instrumentation system,** (AEC Committee on Nuclear Instrument Modules), *AEC Report No. TID-25877*, 45 pages (U.S. Atomic Energy Commission, Washington, D.C., Dec. 1972).

Cotton, I. W., **A position paper—panel session on intelligent terminals—chairman's introduction,** (Proc. 1973 National Computer Conference and Exposition, New York, N.Y., June 4-8, 1973). Paper in *AFIPS Conference Proceedings* **42**, 217-218 (AFIPS Press, Montvale, N.J., 1973).

Coxon, B., Schaffer, R., **Analysis of a cortisol clinical standard by Fourier transform, proton magnetic resonance spectroscopy,** *Anal. Lett.* **6**, No. 4, 291-300 (1973).

Craw, A. R., **A comparison of several methods for forecasting U.S. traffic fatalities,** *NBSIR 73-189*, 45 pages (May 1973). (Available as COM 73-22273 from the National Technical Information Service, Springfield, Va. 22151.)

Craw, A. R., **Analyses for decision in the office of flammable fabrics: the level of the standard,** *NBSIR 73-182*, 90 pages. (Apr. 1973). (Available as COM73-11284 from the National Technical Information Service, Springfield, Va. 22151.)

Craw, A. R., **A summary of the relationships between accident indices and rates following a redefinition of "failure,"** *NBSIR 73-154*, 17 pages (Mar. 1973). (Available as COM 73-10865 from the National Technical Information Service, Springfield, Va. 22151.)

Craw, A. R., Ku, R., **The sensitivity of certain indices associated with two-vehicle accidents,** *Accident Anal. Prev.* **5**, 137-146 (1973).

Craw, A. R., **Time series forecasting of highway accident fatalities,** *NBSIR 73-138*, 37 pages (Mar. 1973). (Available as COM 73-10868 from the National Technical Information Service, Springfield, Va. 22151.)

Creitz, E. C., **Extinction of fires by halogenated compounds—a suggested mechanism,** *Fire Technol.* **8**, No. 2, 131-141 (May 1972).

Creitz, E. C., **Gas chromatographic determination of composition profiles of stable species around a propane diffusion flame,** *J. Chromatogr. Sci.* **10**, 168-173 (Mar. 1972).

Creswell, R. A., Lafferty, W. J., **Microwave spectrum, dipole moment, and conformation of 3,6-dioxabicyclo[3.1.0]hexane,** *J. Mol. Spectrosc.* **46**, No. 3, 371-380 (June 1973).

Cruz, J. E., Rogers, E. H., Hiester, A. E., **Continuous liquid level measurements with time-domain reflectometry,** (Proc. 1972 Cryogenic Engineering Conf., Boulder, Colo., Aug. 9-11, 1972), Paper H-4 in *Advances in Cryogenic Engineering* **18**, 323-327 (Plenum Press, New York, N.Y., 1973).

Currie, L. A., **On the use of small calculators having stacked registers,** *Anal. Lett.* **6**, No. 9, 847-864 (1973).

Currie, L. A., **The limit of precision in nuclear and analytical chemistry,** *Nucl. Instrum. Methods* **100**, 387-395 (1972).

Daney, D. E., McConnell, P. M., Strobridge, T. R., **Low-temperature nitrogen ejector performance,** (Proc. 1972 Cryogenic Engineering Conf., Boulder, Colo., Aug. 9-11, 1972), Paper L-4 in *Advances in Cryogenic Engineering* **18**, 476-485 (Plenum Press, New York, N.Y., 1973).

Danielson, S. L., Howe, D. A., **Use of the television vertical interval to broadcast time for everyone and program captions for the deaf,** *Commun. Soc.* **11**, No. 5, 3-6 (Nov. 1973).

Danos, M., **Baryon resonances in nuclei,** *J. Phys.* **33**, Supplement to No. 8-9, 171-182 (Aug.-Sept. 1972).

Davis, G. T., Eby, R. K., **Glass transition of polyethylene: Volume relaxation,** *J. Appl. Phys.* **44**, No. 10, 4274-4281 (Oct. 1973).

Davis, R. M., **Automation technology for the retail industry (welcome address),** *Proc. Conf. Automation Technology for the Retail Industry, held at the National Bureau of Standards, Gaithersburg, Md., Mar. 15, 1972*, COM-72-10495, pp. 1-7 (National Technical Information Service, Springfield, Va., 1972).

Dellepiane, G., Gussoni, M., Hougen, J. T., **Hamiltonian, symmetry group, and vibrational coordinates for the nonrigid molecule $CXY_2 - C \equiv C - CXY_2$,** *J. Mol. Spectrosc.* **47**, No. 3, 515-530 (Sept. 1973).

Dewitt, D. P., Richmond, J. C., **Theory and measurement of the thermal radiation properties of metals,** Chapter 1 in *Measurement of Physical Properties: Some Special Properties*, E. Passaglia, Ed., **VI**, Part 1, 1-90 (Interscience Publ., New York, N.Y., 1972).

Dhez, P., Ederer, D. L., **Photoionization resonance profile parameters of the $3s3p\ ^1P_1$ two-electron excitation in He I,** *J. Phys. B: At. Mol. Phys.* Letter to Editor **6**, L59-L64 (Apr. 1973).

Dibeler, V. H., Walker, J. A., McCulloh, K. E., **Observations on hot bands in the molecular and dissociative photoionization of acetylene and the heat of formation of the ethynyl ion,** *J. Chem. Phys.* **59**, No. 5, 2264-2268 (Sept. 1, 1973).

Diller, D. E., Sarkes, L. A., **Properties data for LNG,** *Amer. Gas Ass. Mon.* **55**, No. 9, 27-28 (Sept. 1973).

Dillon, T. A., Stephenson, J. C., **Effect of the straight path approximation and exchange forces on vibrational energy transfer,** *J. Chem. Phys.* **58**, No. 9, 3849-3854 (May 1, 1973).

Dodge, M. J., Malitson, I. H., Mahan, A. I., **A special method for precise refractive index measurement for uniaxial optical media,** *Appl. Opt.* **8**, No. 8, 1703-1705 (Aug. 1969).

Donaldson, J. R., **On uncertainty in mass measurement,** *NBSIR 73-151*, 10 pages (Mar. 1973). (Available as COM 73-10866 from the National Technical Information Service, Springfield, Va. 22151.)

Drummond, D., Gallagher, A., **A low resolution scanning multiple Fabry-Perot spectrometer,** *Rev. Sci. Instrum.* **44**, No. 4, 396-399 (Apr. 1973).

Edelsack, E. A., Kropschot, R. H., Olien, N. A., Olsen, J. L., **A Directory of European Low Temperature Research,** *Cryogenics* **13**, No. 3, 132-133 (Mar. 1973).

Ederer, D. L., **Cross-section profiles of resonances in the photoionization continuum of krypton and xenon (600-400 Å),** *Phys. Rev. A* **4**, No. 6, 2263-2270 (Dec. 1971).

Ederer, D. L., Lucatorto, T., Madden, R. P., **Resonances in the photoionization continuum of lithium I (55 to 70 eV),** *J. Phys.* **32**, Supplement to No. 10, C4-85—C4-87 (Oct. 1971).

Ehlers, V. J., Gallagher, A., **Electron excitation of the calcium 4227-Å resonance line,** *Phys. Rev. A* **7,** No. 5, 1573-1585 (May 1973).

Ehrlich, M., **Influence of size of CaF_2:Mn thermoluminescence dosimeters on ^{60}Co gamma-ray dosimetry in extended media,** *Proc. 3d Int. Conf. on Luminescence Dosimetry, held at the Danish AEC Research Establishment Risø, Roskilde, Denmark, Oct. 11-14, 1971,* Risø Report No. 249, Part II, 550-560 (Dec. 1971).

Eisen, H., Rosenstein, M., Silverman, J., **Electron dosimetry using Chalkley-McLaughlin dye-cyanide thin films,** (Proc. Int. Symp. on Dosimetry in Agriculture, Industry, Biology and Medicine, Vienna, Austria, April 1972), Paper in *Dosimetry in Agriculture, Industry, Biology and Medicine,* pp. 615-625 (International Atomic Energy Agency, Vienna, Austria, 1973).

Eisenhower, E. H., **Safety recommendations in ANSI documents,** (Proc. Symp. on Radiation Safety and Protection in Industrial Applications, Washington, D.C., Aug. 16-18, 1972), *DHEW Pub.* No. (FDA) 73-8012, H. F. Klein, Ed., pp. 240-252 (U.S. Department of Health, Education, and Welfare, Public Health Service, Food and Drug Administration, Bureau of Radiological Health, Rockville, Md., Oct. 1972).

Ely, J. F., Hanley, H. J. M., Straty, G. C., **Analysis of the pressure virials and Clausius-Mossotti function for polyatomic gases,** *J. Chem. Phys.* **59,** No. 2, 842-848 (July 15, 1973).

Etz, E. S., Robinson, R. A., Bates, R. G., **Dissociation constant of protonated tris(hydroxymethyl)aminomethane in N-methylpropionamide and related thermodynamic quantities from 10 to 55 °C,** *J. Solution Chem.* **2,** No. 4, 405-415 (1973).

Evans, A. G., **Strength degradation by projectile impacts,** *J. Amer. Ceram. Soc.* **56,** No. 8, 405-409 (Aug. 1973).

Evans, J. P., **High temperature platinum resistance thermometry,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry,* H. H. Plumb, Editor-in-Chief, **4,** Part 2, 899-906 (Instrument Society of America, Pittsburgh, Pa., 1972).

Fabry, A., Jenkins, J. D., **Wall return neutron fluxes for high- and intermediate-energy cavity neutron sources,** *Trans. Amer. Nucl. Soc.* **15,** No. 2, 975-976 (Nov. 1972).

Falge, R. L., Jr., Swartzendruber, L. J., **Influence of clustering on the paramagnetic behavior of a Cu-Ni alloy,** *Phys. Lett.* **44A,** No. 4, 285-286 (June 18, 1973).

Fatiadi, A. J., **Facile coupling of sterically hindered 2,6-dialkylphenols with periodic acid,** *Synthesis Commun. No. 6,* 357-358 (June 1973).

Fatiadi, A. J., **Mechanism of formation of tris(phenylhydrazones) on treatment of cyclohexane-1,3-diones with phenylhydrazine,** *Chem. Ind.*, pp. 38-40 (Jan. 6, 1973).

Feldman, A., Horowitz, D., Waxler, R. M., **Laser damage in materials,** *NBSIR 73-119.* 52 pages. (Feb. 1973). (Available as AD757789 from the National Technical Information Service, Springfield, Va. 22151.)

Ferguson, J. B., **Summary of flame spread and smoke generation tests conducted for operation BREAKTHROUGH,** *NBSIR 73-228,* 27 pages. (July 1973). (Available as PB 222425 from the National Technical Information Service, Springfield, Va. 22151.)

Fey, L., **A time code for the Omega worldwide navigation system,** *Proc. IEEE* **60,** No. 5, 630 (May 1972).

Fickett, F. R., **Magnetoresistivity of copper and aluminum at cryogenic temperatures,** *(Proc. 4th Int. Conf. on Magnet Technology, Brookhaven National Laboratory, Upton, N.Y., Sept. 19-22, 1972),* pp. 539-541 (Atomic Energy Commission, Washington, D.C., 1972).

Fickett, F. R., **Properties of nonsuperconducting technical solids at low temperatures,** *Proc. 4th Int. Conf. on Magnet Technology,* Brookhaven National Laboratory, Upton, N.Y., Sept. 19-22, 1972, pp. 498-516 (Atomic Energy Commission, Washington, D.C., 1972).

Field, R. W., Tilford, S. G., Howard, R. A., Simmons, J. D., **Fine structure and perturbation analysis of the $a^3\Pi$ state of CO,** *J. Mol. Spectrosc.* **44,** No. 2, 347-382 (Nov. 1972).

Field, R. W., Wicke, B. G., Simmons, J. D., Tilford, S. G., **Analysis of perturbations in the $a^3\Pi$ and $A^1\Pi$ states of CO,** *J. Mol. Spectrosc.* **44,** No. 2, 383-399 (Nov. 1972).

Finkel, P. W., Miller, T. R., **Executive summary of a proficiency test assessment of clinical laboratory capability in the United States,** *NBSIR 73-162,* 9 pages (May 1973).

Finkel, P. W., Rowen, J. W., **Clinical laboratory performance analysis using proficiency test statistics,** *NBSIR 73-197,* 61 pages, (June 1973). (Available as COM 73-11253 from the National Technical Information Service, Springfield, Va. 22151.)

Finnegan, T. F., Witt, T. J., Field, B. F., Toots, J., **Measurements of $2e/h$ via the AC Josephson effect,** (Proc. 4th Int. Conf. on Atomic Masses and Fundamental Constants. Teddington, England, Sept. 6-10, 1971), Paper in *Atomic Masses and Fundamental Constants* **4,** J. H. Sanders and A. H. Wapstra, Eds., pp. 403-410 (Plenum Press, New York, N.Y., 1972).

Flynn, D. R., **Inter-Noise 72 Panel: Noise measurement—problems, perils and pitfalls,** *Noise/News* **2,** No. 1, 7-9 (Jan.-Feb. 1973).

Flynn, H. G., Cook, R. K., **The effects of sonic boom and similar impulsive noise on structures,** *Environmental Protection Agency Report No. NTID200.12,* 19 pages (available as PB206725 from the National Technical Information Service, Springfield, Va. 22151, Dec. 31, 1971).

Flynn, J. H., **Instrumental limitations upon the measurement of temperature and rate of energy production by differential scanning calorimetry,** (Proc. 3d Int. Conf. on Thermal Analysis, Davos, Switzerland, Aug. 23-28, 1971), Paper in *Thermal Analysis,* H. G. Wiedemann, Ed. **1,** 127-138 (Birkhäuser Verlag, Basel, Switzerland, 1972).

Fraker, A. C., Ruff, A. W., **The effect of solution pH on the saline water corrosion of titanium alloys,** (Proc. 2nd Int. Conf. on Titanium, Boston, Mass., May 2-5, 1972), Paper in *Titanium Science and Technology,* R. I. Jaffee and H. M. Burte, Eds., **4,** 2655-2663 (Plenum Publishing Corp., New York, N.Y., 1973).

Fraker, A. C., Ruff, A. W., Yeager, M. P., **Corrosion of titanium alloys in physiological solutions,** (Proc. 2nd Int. Conf. on Titanium, Boston, Mass., May 2-5, 1972), Paper in *Titanium Science and Technology,* R. I. Jaffee and H. M. Burte, Eds., **4,** 2447-2457 (Plenum Publishing Corp., New York, N.Y., 1973).

Franzen, D. L., **Continuous laser-sustained plasmas,** *J. Appl. Phys.* **44,** No. 4, 1727-1732 (Apr. 1973).

Frederikse, H. P. R., Hosler, W. R., **Electrical conductivity of coal slag,** *J. Amer. Ceram. Soc.* **56,** No. 8, 418-419 (Aug. 1973).

Freeman, D. H., Kuehner, E. C., **Laser detection of small particles in liquids,** *Annals N.Y. Acad. Sci.* **158,** Article 3, 731-740 (June 20, 1969).

Frenkiel, F. N., Klebanoff, P. S., **Probability distributions and correlations in a turbulent boundary layer,** *Phys. Fluids* **16,** No. 6, 725-737 (June 1973).

Fröhlich, C., Geist, J., Kendall, J., Sr., Marchgraber, R. M., **The Third International Comparisons of Pyrheliometers and a comparison of radiometric scales,** *Sol. Energy* **14,** 157-166 (1973).

Fromhold, A. T., Jr., Kruger, J., **Space-charge and concentration-gradient effects on anodic oxide film formation,** *J. Electrochem. Soc.* **120,** No. 6, 722-729 (June 1973).

Frommer, M. A., Murday, J. S., Messalem, R. M., **Solubility and diffusivity of water and of salts in an aromatic polyamide film,** *Eur. Polym. J.* **9,** 367-373 (1973).

Fuller, E. G., **Photonuclear Physics 1973. Where we are and how we got there,** (Proc. Int. Conf. on Photonuclear Reactions and Applications, Pacific Grove, Calif., Mar. 26-30, 1973), Paper in *International Conference on Photonuclear Reactions and Applications*, B. L. Berman, Ed., pp. 1201-1224 (Ernest O. Lawrence Livermore Laboratory, University of California, Livermore, Calif., 1973). (Available as CONF-730301 from the National Technical Information Service, Springfield. Va. 22151).

Fuller, G. H., **Problems of compiling and evaluating or what is the nuclear moment?** (Proc. Int. Conf. on Nuclear Moments and Nuclear Structure, Osaka, Japan, Sept. 4-8, 1972), *J. Phys. Soc. Jap.* **34,** Supplement, 153-154 (1973).

Furukawa, G. T., Bigge, W. R., Riddle, J. L., **Triple point of argon,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4,** Part 1C. 231-243 (Instrument Society of America, Pittsburgh, Pa., 1972).

Furukawa, G. T., Reilly, M. L., **Application of precise heat-capacity data to the analysis of the temperature intervals of the international practical temperature scale of 1968 in the region of 90 K,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4,** Part 1A, 27-36 (Instrument Society of America, Pittsburgh, Pa., 1972).

Furukawa, G. T., Riddle, J. L., Bigge, W. R., **Investigation of freezing temperatures of National Bureau of Standards tin standards,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4,** Part 1D, 247-263 (Instrument Society of America, Pittsburgh, Pa., 1972).

Furukawa, G. T., **Vapor pressures of ^{20}Ne and ^{22}Ne,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4,** Part 1B. 127-135 (Instrument Society of America, Pittsburgh, Pa., 1972).

Gadzuk, J. W., Lucas, A. A., **Field-emission tails and tunneling lifetimes,** *Phys. Rev. B* **7,** No. 11, 4770-4775 (June 1, 1973).

Gadzuk, J. W., Plummer, E. W., **Field emission energy distribution (FEED),** *Rev. Mod. Phys.* **45,** No. 3, 487-548 (July 1973).

Gans, W. L., Nahman, N. S., **Fast Fourier transform implementation for the calculation of network frequency domain transfer functions from time domain waveforms,** *NBSIR 73-303*, 53 pages (Dec. 1972). (Available as COM 73-10509 from the National Technical Information Service, Springfield. Va. 22151.)

Garvin, D., **Chemical kinetics data survey IV. Preliminary tables of chemical data for modelling of the stratosphere,** *NBSIR-203*, 95 pages, (May 1973). (Available as COM73-11288 from the National Technical Information Service, Springfield. Va. 22151.)

Garvin, D., **Chemical kinetics data survey V. Sixty-six contributed rate and photochemical data evaluations on ninety-four reactions,** *NBSIR 73-206*, 121 pages, (May 1973). (Available as COM 73-11262 from the National Technical Information Service, Springfield, Va. 22151.)

Garvin, D., Hampson, R. F., **Atmospherical modelling and the chemical data problem,** *Proc. AIAA/AMS Int. Conf. on the Environmental Impact of Aerospace Operations in the High Atmosphere, Denver, Colo., June 11-13, 1973*, AIAA Paper No. 73-500. 1-6 (American Institute of Aeronautics and Astronautics, New York, N.Y., 1973).

Garvin, D., Hampson, R. F., **Evaluated rate and photochemical data for modeling of the stratosphere,** (Proc. 2d Conf. on Climatic Impact Assessment Program, Cambridge, Mass., Nov. 14-17, 1972), *DOT Report* No. TSC-OST-73-4, 12 pages (Transportation Systems Center, Department of Transportation, Cambridge, Mass., 1972).

Garvin, D., **Progress in rate evaluations,** (Proc. 7th Thermochemistry Working Group, Interagency Chemical Rocket Propulsion Group, Cleveland, Ohio, Apr. 9-11, 1969), *CPIA Pub. 189*. **I,** 109-118 (Chemical Propulsion Information Agency, Johns Hopkins University, Applied Physics Laboratory, Silver Spring, Md., Aug. 1969).

Geist, J., **New NBS scale of irradiance,** *Appl. Opt.* **12,** No. 4, 907-908 (Apr. 1973).

Geist, J., **Theoretical analysis of laboratory blackbodies. 1: A generalized integral equation,** *Appl. Opt.* **12,** No. 6, 1325-1330 (June 1973).

Geltman, S., **Free-free radiation in electron-neutral atom collisions,** *J. Quant. Spectrosc. Radiat. Transfer* **13,** 601-613 (June 1973).

Giarrantano, P. J., Hess, R. C., Jones, M. C., **Forced convection heat transfer to subcritical Helium I,** *NBSIR 73-322*, 43 pages (May 1973). (Available as COM 73-11464-AS from the National Technical Information Service, Springfield. Va. 22151.)

Gilman, F. J., Kugler, M., Meshkow, S., **Pionic transitions as tests of the connection between current and constituent quarks,** *Phys. Lett.* **45B,** No. 5, 481-486 (Aug. 20, 1973).

Ginnings, D. C., Reilly, M. L., **Calorimetric measurement of thermodynamic temperatures above 0 °C using total blackbody radiation,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4,** Part 1, 339-348 (Instrument Society of America, Pittsburgh, Pa., 1972).

Gonano, R., **Isothermal comparisons of He^3 and He^4 vapor pressures,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4**, Part 1, 121-126 (Instrument Society of America, Pittsburgh, Pa., 1972).

Grabner, L., Wong, E. Y., **Zeeman effect of no-phonon $^4A_{2g}$ - $^4T_{2g}$ transition of Cr^{3+} in TiO_2,** *Phys. Rev. B* **8**, No. 3, 1032-1037 (Aug. 1, 1973).

Greenspan, M., **Transducer measurements: Use of the current probe,** *J. Acoust. Soc. Amer. Tech. Notes and Res. Briefs* **53**, No. 4, 1186-1187 (Apr. 1973).

Grundl, J. A., **Brief review of integral measurements with fission spectrum neutrons,** (Proc. of the Consultants' Meeting, Vienna, Austria, Aug. 1971), Paper in *Prompt Fission Neutron Spectra*, pp. 29-32 (International Atomic Energy Agency, Vienna, Austria, 1972).

Grundl, J. A., **Measurement of the average fission cross-section ratio, $\sigma_f(^{235}U)/\sigma_f(^{238}U)$ for ^{235}U and ^{239}Pu fission neutrons,** (Proc. Specialists's Meeting on Prompt Fission Neutron Spectra, Vienna, Austria, Aug. 1971), Paper in *Prompt Fission Neutron Spectra*, pp. 107-111 (International Atomic Energy Agency, Vienna, Austria, 1972).

Grundl, J. A., Spiegel, V., Jr., Eisenhauer, C., **Measurement of ^{235}U and ^{238}U fission cross sections for ^{252}Cf spontaneous fission neutrons,** *Trans. Amer. Nucl. Soc.* **15**, No. 2, 945-946 (1972).

Guildner, L. A., Anderson, R. L., Edsinger, R. E., **Effects of sorption on the realization of the thermodynamic scale,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4**, Part 1E, 313-322 (Instrument Society of America, Pittsburgh, Pa., 1972).

Haar, L., **The ideal gas-calorimetric thermometer,** *Science* **176**, 1293-1296 (June 23, 1972).

Hall, J. L., Bordé, C., **Measurement of methane hyperfine structure using laser saturated absorption,** *Phys. Rev. Lett.* **30**, No. 22, 1101-1104 (May 28, 1973).

Hallowell, P. L., Bertozzi, W., Heisenberg, J., Kowalski, S., Maruyama, X., Sargent, C. P., Turchinetz, W., Williamson, C. F., Fivozinsky, S. P., Lightbody, J. W., Jr., Penner, S., **Electron scattering from ^{19}F and ^{40}Ca,** *Phys. Rev. C* **7**, No. 4, 1396-1409 (Apr. 1973).

Halsey, N., Mitchell, R. A., Mordfin, L., **Evaluation of GRP rod and rope materials and associated end fittings,** *NBSIR 73-129*, 87 pages (Dec. 1972). (Available as AD 759931 from the National Technical Information Service, Springfield, Va. 22151.)

Hamilton, C. A., **Analog-computer studies of mixing and parametric effects in Josephson junctions,** *J. Appl. Phys.* **44**, No. 5, 2371-2377 (May 1973).

Hampson, R. F., Garvin, D., Herron, J. T., Huie, R. E., Kurylo, M. J., Laufer, A. H., Okabe, H., Scheer, M. D., Tsang, W., **Chemical kinetics data survey VI: Photochemical and rate data for twelve gas phase reactions of interest for atmospheric chemistry,** *NBSIR 73-207*, 127 pages (Aug. 1973). (Available as AD 769266 from the National Technical Information Service, Springfield, Va. 22151.)

Hartman, A. W., Rosberry, F. W., Simpson, J. A., **A non-contacting length comparator with 10 nanometer precision,** *Opt. Eng.* **12**, No. 3, 95-101 (May/June 1973).

Hastie, J. W., **Mass spectrometric analysis of 1 atm flames: Apparatus and the $CH_4 - O_2$ system,** *Combust. Flame* **21**, 187-194 (1973).

Hastie, J. W., **Mass spectrometric studies of flame inhibition: Analysis of antimony trihalides in flames,** *Combust. Flame* **21**, 49-54 (1973).

Hauge, R. H., Margrave, J. L., Hastie, J. W., **Infrared spectra of matrix isolated ZrF_2, ZrF_3, and ZrF_4,** *High Temp. Sci.* **5**, 89-96 (1973).

Heinrich, K. F. J., **Errors in electron probe microanalysis,** *Proc. of the Seminar on Quantitative Analysis with Electron Microprobes and Secondary Ion Mass Spectrometry, Jülich, Germany, Oct. 18-20, 1972*, pp. 68-79 (Mar. 1973).

Heinrich, K. F. J., **The application of Monte-Carlo calculations in electron probe microanalysis,** *Proc. of the Seminar on Quantitative Analysis with Electron Microprobes and Secondary Ion Mass Spectrometry, Jülich, Germany, Oct. 18-20, 1972*, pp. 149-155 (Mar. 1973).

Hessel, M. M., Lucatorto, T. B., **The rotating heat-pipe oven; a universal device for the containment of atomic and molecular vapors,** *Rev. Sci. Instrum.* **44**, No. 5, 561-563 (May 1973).

Hjortenberg, P. E., McLaughlin, W. L., **Use of radiochromic dye systems for dosimetry,** (Proc. Regional Conf. on Radiation Protection, Jerusalem, Israel, Mar. 1973), Paper in *Radiation Protection*, Y. Feige and T. S. Schlesinger, Eds., **1**, 122-140 (Israel Atomic Energy Commission, Yavne, Israel, 1973).

Hord, J., **Cavitation in liquid cryogens. III – Ogives,** *NASA CR-2242*, 235 pages (National Aeronautics and Space Administration, Washington, D.C., May 1973).

Hougen, J. T., **Tabulation of hyperfine splittings in rotational F_1 and F_2 levels of the ground vibrational state of $^{12}CH_4$, for $J \leq 20$,** *J. Mol. Spectrosc.* **46**, No. 3, 490-501 (June 1973).

Hsieh, C., Thomson, R., **Lattice theory of fracture and crack creep,** *J. Appl. Phys.* **44**, No. 5, 2051-2063 (May 1973).

Hudson, R. P., **Book: Principles and application of magnetic cooling,** *Series in Low Temperature Physics* **2**, 1-230 (North-Holland Publishing Co., Amsterdam, The Netherlands, 1972).

Hudson, R. P., Pfeiffer, E. R., **Temperature scale for cerous magnesium nitrate,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4**, Part 2, 1279-1285 (Instrument Society of America, Pittsburgh, Pa., 1972).

Huggett, C., **Carpet flammability and the NBS corridor fire program,** *ASTM Stand. News* **1**, No. 5, 16-20 (May 1973).

Hummer, D. G., Kunasz, C. V., Kunasz, P. B., **Numerical evaluation of the formal solution of radiative transfer problems in spherical geometries,** *Comput. Phys. Commun.* **6**, 38-57 (1973).

Hust, J. G., Clark, A. F., **A survey of compatibility of materials with high pressure oxygen service,** *Cryogenics* **13**, No. 6, 325-336 (June 1973).

Hust, J. G., Powell. R. L., Sparks, L. L., **Methods for cryogenic thermocouple thermometry,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4**, Part 3. 1525-1535 (Instrument Society of America, Pittsburgh, Pa., 1972).

Jacox. M. E., Milligan, D. E., **Matrix isolation study of the vacuum-ultraviolet photolysis of methanol. The infrared spectrum of the CH_2OH free radical,** *J. Mol. Spectrosc.* **47**, No. 1, 148-162 (July 1973).

Jespersen, J. L., Blair. B. E., Gatterer. L. E., **Characterization and concepts of time-frequency dissemination,** *Proc. IEEE* **60**, No. 5, 502-521 (May 1972).

Jespersen, J. L., Fey, L., **"Time-telling" techniques,** *IEEE Spectrum* **9**, No. 5. 51-58 (May 1972).

Jones. F. E., **Air density and helicopter lift,** *Joint Army-Navy Aircraft Instrumentation Research. JANAIR Report No. 721201*, 54 pages (Available as AD754420 from the National Technical Information Service, Springfield. Va. 22151, Jan. 1973).

Jones. M. C., **Use of superconductors in thermal insulations,** *Cryogenics* **13**, No. 2. 83-84 (Feb. 1973).

Judd. D. B., Eastman, A. A., **Prediction of target visibility from the colors of target and surround,** *Illum. Eng.* **66**, No. 4. 256-266 (Apr. 1971).

Kamper. R. A., Simmonds. M. B., Adair. R. T., Hoer. C. A., **Quantum mechanical measurement of rf attenuation,** *Proc. 1972 Applied Superconductivity Conf., Annapolis. Md., May 1-3. 1972*, pp. 696-700 (IEEE. Inc., New York. N.Y., 1972).

Kamper, R. A., **Survey of noise thermometry,** (Proc. 5th Symp. on Temperature. Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4**, Part 1, 349-354 (Instrument Society of America, Pittsburgh, Pa., 1972).

Kaufman, V., Sugar. J., **One-electron spectrum of singly ionized ytterbium (Yb II),** *J. Opt. Soc. Amer.* **63**, No. 9. 1168-1172 (Sept. 1973).

Keplinger. M. S., **The case for invisible copies,** *Proc. 33d Annual Meeting of the American Society for Information Science, Philadelphia, Pa., Oct. 11-15, 1970.* **7**, 241-243 (1970).

Kidnay, A. J., Hiza, M. J., Miller. R. C., **Liquid-vapour equilibria research on systems of interest in cryogenics—A survey,** *Cryogenics* **13**, No. 10. 575-599 (Oct. 1973).

Kirby, R. K., **Thermal expansion of tungsten from 293 to 1800 K,** *High Temp.-High Pressures* **4**, 459-462 (1972).

Kirchhoff, W. H., Lide, D. R., Jr., **Microwave spectrum, dipole moment, and quadrupole coupling constant of trifluoramine oxide,** *J. Chem. Phys.* **51**, No. 1, 467-468 (July 1, 1969).

Kirchhoff, W. H., **Microwave spectroscopy—a molecular probe,** *Ind. Res.* **12**, No. 2, 38-41 (Feb. 1970).

Kirchhoff, W. H., **Microwave spectroscopy,** *Chem. Eng. News* **47**, 88-98 (Mar. 24, 1969).

Klein, W., **Perturbation solution of the Kirkwood-Salsburg equation,** *J. Math. Phys.* **14**, No. 8, 1049-1059 (Aug. 1973).

Koonce, C. S., **Enhancement effects: Theory,** (Proc. of a Summer Course on The Science and Technology of Superconductivity, held at Georgetown University, Washington, D.C., Aug. 13-26, 1971), Paper in *The Science and Technology of Superconductivity*, W. D. Gregory, W. N. Mathews, Jr., and E. A. Edelsack, Eds., **1**, 389-403 (Plenum Press, New York. N.Y., 1973).

Koonce, C. S., **Theory of superconducting semiconductors,** (Proc. of a Summer Course on The Science and Technology of Superconductivity, held at Georgetown University, Washington, D.C., Aug. 13-26, 1971). Paper in *The Science and Technology of Superconductivity*, W. D. Gregory, W. N. Mathews, Jr., and E. A. Edelsack, Eds., **1**, 373-387 (Plenum Press. New York. N.Y., 1973).

Krauss. M., Julienne, P. S., **Dissociative recombination of $e + CH^+ (X\ ^1\Sigma^+)$,** *Astrophys. J.*, **183**, L139-L141 (Aug. 1, 1973).

Kruger. J., Ambrose, J. R., **The Role of passive film growth kinetics and properties in stress corrosion and crevice corrosion susceptibility,** *NBSIR 73-244*, 75 pages (July 1973). (Available as AD 767326 from the National Technical Information Service, Springfield. Va. 22151.)

Kruger, J., **Application of ellipsometry to electrochemistry,** Chapter in *Advances in Electrochemistry and Electrochemical Engineering*, R. H. Muller, Ed., **9**, 227-280 (John Wiley & Sons. Inc., New York. N.Y. 1973).

Ku. H. H., **Analysis of information-an alternative approach to the detection of a correlation between the sexes of adjacent sibs in human families,** *Biometrics* **27**, 175-182 (Mar. 1971).

Ku. H. H., Kullback. S., **Analysis of multidimensional contingency tables: An information theoretic approach,** *Contributed Papers 37th Session of the International Statistical Institute, London, England, Sept. 3-11, 1969*, pp. 156-158 (Sept. 11, 1969).

Ku, H. H., Varner. R., Kullback. S., **Analysis of multidimensional contingency tables,** *Proc. 14th Conf. on the Design of Experiments in Army Research Development and Testing, Edgewood Arsenal, Md., Oct. 23-25, 1968*, ARO-D Report 69-2, pp. 141-180 (1969).

Kuriyama, M., **Dynamical diffraction equations for imperfect crystals,** *Z. Naturforsch.* **28a**, No. 5. 622-626 (1973).

Kushner. L. M., **Progress report on activities of the National Bureau of Standards under the 1967 Amendments to the Flammable Fabrics Act,** *Proc. 3d Annual Meeting of the Information Council on Fabric Flammability, New York, N.Y., Dec. 4, 1969*, pp. 156-163 (Information Council on Fabric Flammability, New York. N.Y., 1969).

Kusuda, T., **Climatological data at the proposed prototype sites in the United States for the evaluation of HUD Operation BREAKTHROUGH housing systems,** *NBSIR 73-144*, 137 pages (Apr. 10. 1973). (Available as PB 220849/4 from the National Technical Information Service, Springfield. Va. 22151.)

LaFleur. P. D., **Retention of mercury when freeze-drying biological materials,** *Anal. Chem.* **45**, No. 8. 1534-1536 (July 1973).

LaFleur. P. D., Thompson. B. A., **Gamma-ray spectroscopy,** Paper in the *Encyclopedia of Chemistry, Third Edition*, C. Hampel and G. G. Hawley, Eds., pp. 1032-1033 (Van Nostrand Reinhold Co., New York. N.Y., 1973).

LaVilla, R. E., **Carbon and fluorine x-ray emission and fluorine *K* absorption spectra of the fluoromethane molecules, $CH_{4-n}F_n$ ($0 \leq n \leq 4$). II,** *J. Chem. Phys.* **58,** No. 9, 3841-3848 (May 1, 1973).

LaVilla, R. E., **$M_{2,3}$-Region x-ray emission spectrum from gaseous krypton,** *Phys. Rev. A.* **8,** No. 2, 1143-1145 (Aug. 1973).

Leasure, W. A., Jr., Bender, E. K., **Tire-road interaction noise,** (Proc. 1973 International Noise Control Engineering Conference, Copenhagen, Denmark, Aug. 22-24, 1973), Paper in *Inter-Noise 73 Proceedings*, O. J. Pedersen, Ed., pp. 421-425 (Inter-Noise 73, Technical University, Lyngby, Denmark, 1973).

Leasure, W. A., Jr., Rubin, A. I., Fath, J. M., Fisher, R. L., Flynn, D. R., **Fundamentals of noise: Measurement, rating schemes, and standards,** *Environmental Protection Agency Report No. NTID300.15*, 163 pages (Available as PB206727 from the National Technical Information Service, Springfield, Va. 22151, Dec. 31, 1971).

Ledbetter, H. M., **Estimation of Debye temperatures by averaging elastic coefficients,** *J. Appl. Phys.* **44,** No. 4, 1451-1454 (Apr. 1973).

Lee, R. D., Kostkowski, H. J., Quinn, T. J., Chandler, P. R., Jones, T. P., Tapping, J., Kunz, H., **Intercomparison of the IPTS 68 above 1064 °C by four national laboratories,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4,** Part 1, 377-393 (Instrument Society of America, Pittsburgh, Pa., 1972).

Lee, T. G., Huggett, C., **Interlaboratory evaluation of the tunnel test (ASTM E 84) applied to floor coverings,** *NBSIR 73-125*, 56 pages, (Mar. 1973). (Available as COM 73-11189 from the National Technical Information Service, Springfield, Va. 22151.)

Le Neindre, B., Tufeu, R., Bury, P., Sengers, J. V., **Thermal conductivity of carbon dioxide and steam in the supercritical region,** *Ber. Bunsenges. Physik. Chem.* **77,** No. 4, 262-275 (1973).

Levine, J., Stebbins, R. T., **Ultrasensitive laser interferometers and their application to problems of geophysical interest,** *Phil. Trans. Roy. Soc. London A* **274,** 279-284 (1973).

Levine, J., Stebbins, R., **Upper limit on the gravitational flux reaching the earth from the Crab pulsar,** *Phys. Rev. D* **5,** No. 7, 1465-1468 (Apr. 1, 1972).

Lew, H. S., **The effect of impact loadings on the performance of wood joist subflooring systems,** *NBSIR 73-187*, 49 pages (May 1973). (Available as PB 221188 from the National Technical Information Service, Springfield, Va. 22151.)

Liebman, J. F., **On the electron affinity of perfluorocycloalkanes and perfluoroalkanes,** *J. Fluorine Chem.* **3,** 27-33 (1973/74).

Liebman, J. F., Vanderspurt, T. H., **Polarization of X-F bonds,** *J. Fluorine Chem.* **2,** 413-424 (1972/73).

Linsky, J. L., **A recalibration of the quiet sun millimeter spectrum based on the moon as an absolute radiometric standard,** *Solar Phys.* **28,** No. 2, 409-418 (Feb. 1973).

Livingston, R. C., Rothschild, W. G., Rush, J. J., **Molecular reorientation in plastic crystals: Infrared and Raman band shape analysis of neopentane,** *J. Chem. Phys.* **59,** No. 5, 2498-2508 (Sept. 1, 1973).

Lovejoy, R. W., Olson, W. B., **Ground state rotational constants for $^{28}SiH_3D$,** *J. Chem. Phys. Letters to Editor* **57,** No. 5, 2224-2225 (Sept. 1, 1972).

Lozier, D. W., Maximon, L. C., Sadowski, W. L., **A bit comparison program for algorithm testing,** *Comput. J.* **16,** No. 2, 111-117 (May 1973).

Key words: Algorithm testing; bit comparison testing; computer algorithm; mathematical function subroutines.

In view of the increasingly important role of the computer in scientific calculations, the development of computer algorithms for elementary and special functions has been given a great deal of attention. The development of algorithms cannot be divorced from their evaluation, for a computer algorithm is judged solely on the basis of its performance characteristics. These include storage requirements, speed and accuracy. The present paper will deal only with the accuracy aspect of algorithm testing. The other two aspects must be evaluated in the context in which the algorithm is used. In this paper by an algorithm we mean a computer algorithm, i.e., an implementation of a mathematical algorithm in a specific environment. The environment is taken to include factors that may affect the algorithm, e.g., the operating system under which the program is run and hardware algorithms for arithmetic operations. Whereas in some instances mathematical algorithms have been successfully used to locate hardware malfunctions that were not traceable by normal trouble shooting tests, any malfunctions of the software or hardware will not be considered here to be part of the environment.

Ludtke, P. R., **Development of insulation transfer-standards using a flat plate calorimeter,** *NBSIR 73-301*, 43 pages (Mar. 1973). (Available as COM 73-10762 from the National Technical Information Service, Springfield, Va. 22151.)

Lyndon, R., McDonough, T., Newman, M., **On products of powers in groups,** *Proc. Amer. Math. Soc.* **40,** No. 2, 419-420 (Oct. 1973).

Key words: Free groups; powers.

In this note we show that a product of Nth powers in a group cannot in general be expressed as a product of fewer Nth powers. This extends a result of Lyndon and Newman.

Mabie, C. P., **Petrographic study of the refractory performance of high-fusing dental alloy investments: II. Silica-bonded investments,** *J. Dent. Res.* **52,** No. 4, 758-773 (1973).

Maienthal, E. J., **Determination of trace elements in metals by differential cathode ray polarography,** *Amer. Lab.*, pp. 25-33 (1972).

Maki, A. G., **Infrared spectra of CS_2: Measurement of "hot" bands associated with the 2325 cm^{-1} and 2962 cm^{-1} bands,** *J. Mol. Spectrosc.* **47,** No. 2, 217-225 (Aug. 1973).

Maki, A. G., Johnson, D. R., **Microwave spectra of carbonyl sulfide: Measurements of ground state and vibrationally excited $^{16}O^{13}C^{33}S$, $^{18}O^{12}C^{32}S$, and other isotopic species,** *J. Mol. Spectrosc.* **47,** No. 2, 226-233 (Aug. 1973).

Maki, A. G., Sams, R. L., Olson, W. B., **Infrared determination of C_0 for phosphine via perturbation-allowed $\Delta|k-l|=\pm 3$ transitions in the $3\nu_2$ band,** *J. Chem. Phys.* **58,** No. 10, 4502-4512 (May 15, 1973).

Mandel, J., **Structure analysis in two-way tables of measurement data,** *Proc. 39th Session of the International Statistical Institute, Vienna, Austria, Aug. 20-30, 1973*, **2,** 697-705 (International Statistical Institute, Vienna, Austria, 1973).

Martin, W. C., **Some aspects of the energy-level structures of lanthanide atoms and ions,** *Opt. Pura Apl.* **5,** No. 3, 181-191 (1972).

Martin, W. C., Sugar, J., **Classifications of the resonance lines of europium III**, *Astrophys. J.* **184**, 671-674

Masters, L. W., Wolfe, W. C., Rossiter, W. J., Jr., Shaver, J. R., **State of the art on durability testing of building components and materials**, *NBSIR 73-132*, 128 pages (Mar. 1973). (Available as PB 222300 from the National Technical Information Service, Springfield, Va. 22151.)

McCamy, C. S., Pope, C. I., **Redox blemishes— their cause and prevention**, (Proc. National Microfilm Association, Boston, Mass., May 7, 1969), *J. Microgr.* **3**, No. 4, 165-170 (June 1970).

McCamy, C. S., Pope, C. I., **The stability of silver-gelatin images**, (Proc. 2d Int. Congress on Reprography, Cologne, Germany, Oct. 25-31, 1967), Paper A3-1 in Section A3 of *Reprographie* **II**, 112-115 (Verlag Dr. Othmar Helwich, Darmstadt, Germany und Wien, Austria, 1969).

McCamy, C. S., **Properties of photographic films related to information recovery**, *SPIE Seminar Proc. on Image Information Recovery, Philadelphia, Pa., Oct. 24-25, 1968*, **16**, 11-19 (Society of Photo-Optical Instrumentation Engineers, Redondo Beach, Calif., 1969).

McCamy, C. S., **The evaluation and manipulation of photographic images**, Chapter in *Picture Processing and Psychopictorics*, A. Rosenfeld and B. S. Lipkin, Eds., 57-74 (Academic Press, Inc., New York, N.Y., 1970).

McClenon, R., **General purpose scientific dociment code operations under Exec 8**, *NBSIR 73-198*, 21 pages. (May 1973). (Available as COM73-11289 from the National Technical Information Service, Springfield, Va. 22151.)

McLaughlin, W. L., Bjergbakke, E., **Electrochemical dosimetry using ferric-fluoride ion complex formation**, (Proc. Int. Symp. on Dosimetry in Agriculture, Industry, Biology and Medicine, Vienna, Austria, Apr. 1972), Paper in *Dosimetry in Agriculture, Industry, Biology and Medicine*, pp. 383-396 (International Atomic Energy Agency, Vienna, Austria, 1973).

McLaughlin, W. L., Hjortenberg, P. E., Radak, B. B., **Absorbed-dose measurements with thin films**, (Proc. Int. Symp. on Dosimetry in Agriculture, Industry, Biology and Medicine, Vienna, Austria, April 1972), Paper in *Dosimetry in Agriculture, Industry, Biology and Medicine*, pp. 577-597 (International Atomic Energy Agency, Vienna, Austria, 1973).

McLaughlin, W. L., Kosanić, M., Draganić, I. G., **Extending the response of the pararosaniline dye system**, (Proc. Int. Symp. on Dosimetry in Agriculture, Industry, Biology and Medicine, Vienna, Austria, Apr. 1972), Paper in *Dosimetry in Agriculture, Industry, Biology and Medicine*, pp. 362-366 (International Atomic Energy Agency, Vienna, Austria, 1973).

Meijer, P. H. E., **Calculations on the specific heat and the susceptibility of interacting spin systems**, (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4**, Part 1, 1267-1278 (Instrument Society of America, Pittsburgh, Pa., 1972).

Meinke, W. W., **Is radiochemistry the ultimate in trace analysis?**, *Pure Appl. Chem.* **34**, 93-104 (1973).

Mielenz, K. D., Eckerle, K. L., Madden, R. P., Reader, J., **New reference spectrophotometer**, *Appl. Opt.* **12**, No. 7, 1630-1641 (July 1973).

Mihalas, D., Hummer, D. G., **Analyses of light-ion spectra in stellar atmospheres. III. Nitrogen III in the O stars**, *Astrophys. J.* **179**, No. 3, 827-845 (Feb. 1, 1973).

Miller, G. K., Goodman, K. M., **The Shirley Highway Express-Bus-on Freeway demonstration project—first year results**, *DOT/UMTA* **2**, 120 pages (Urban Mass Transportation Administration, Department of Transportation, Washington, D.C., Nov. 1972). (Available as PB 214333 from the National Technical Information Service, Springfield, Va. 22151).

Miller, G., **The Shirley Highway Express-Bus-on Freeway demonstration project—Project description**, *Report DOT/UMTA* **1**, 87 pages (Urban Mass Transportation Administration, Department of Transportation, Washington, D.C., Aug. 1971). (Available as PB 218983 from the National Technical Information Service, Springfield, Va. 22151).

Milligan, D. E., Jacox, M. E., Guillory, W. A., **Infrared spectrum on the NO_2^- ion isolated in an argon matrix**, *J. Chem. Phys.* **52**, No. 8, 3864-3868 (Apr. 15, 1970).

Milligan, D. E., Jacox, M. E., **Infrared and ultraviolet spectroscopic studies of a number of small free radicals and molecular ions in a matrix environment**, (Proc. Symp. on Spectroscopic Methods in Cryochemistry and of the Chemistry of High Temperature Species, Minneapolis, Minn., Apr. 1969), Paper in *Advances in High Temperature Chemistry*, Leroy Eyring, Ed. **4**, 1-42 (Academic Press, Inc., New York, N.Y., 1971).

Milligan, D. E., Jacox, M. E., **Infrared spectroscopic evidence for the stabilization of HAr_2^+ in solid argon at 14 K**, *J. Mol. Spectrosc.* **46**, No. 3, 460-469 (June 1973).

Milligan, D. E., Jacox, M. E., **Matrix isolation spectroscopy**, Chapter 1 in *MTP (Medical and Technical Publishing) International Review of Science. Physical Chemistry Series One. Vol. 3. Spectroscopy*, D. A. Ramsay, Ed., pp. 1-32 (Butterworth & Co., Ltd., London, England and University Park Press, Baltimore, Md., 1972).

Mitchell, R. A., Woolley, R. M., Chwirut, D. J., **Composite-overlay reinforcement of cutouts and cracks in metal sheet**, *NBSIR 73-201*, 96 pages, (Feb. 1973). (Available as COM73-11121 from the National Technical Information Service, Springfield, Va. 22151.)

Mittag, K., **Kapitza conductance and thermal conductivity of copper niobium and aluminum in the range from 1.3 to 2.1 K**, *Cryogenics* **13**, No. 2, 94-99 (Feb. 1973).

Molino, J., **Perceiving the range of a sound source when the direction is known**, *J. Acoust. Soc. Amer.* **53**, No. 5, 1301-1304 (May 1973).

Morris, J. C., Walker, J. H., **Electron-neutral transport cross section of mercury**, *J. Appl. Phys.* **44**, No. 10, 4558-4561 (Oct. 1973).

Mulholland, G. W., **Line of symmetry for the classical equation of state**, *J. Chem. Phys.* **59**, No. 5, pp. 2738-2741 (Sept. 1, 1973).

Molino, J. A., **Pure-tone equal-loudness contours for standard tones of different frequencies**, *Perception Psychophysics* **14**, No. 1, 1-4 (1973).

Moore, R. T. **Penetration resistance tests of reinforced concrete barriers**, *NBSIR 73-101*, 83 pages (Dec. 1972). (Available as COM 73-10867 from the National Technical Information Service, Springfield, Va. 22151.)

Moore, R. T., **Penetration tests on J-SIIDS barriers,** *NBSIR 73-223*, 86 pages, (June 4, 1973). (Available as COM 73-11120 from the National Technical Information Service, Springfield, Va. 22151.)

Mopsik, F. I., Broadhurst, M. G., **Grüneisen constants of polymers,** *J. Appl. Phys.* **44,** No. 10, 4261-4264 (Oct. 1973).

Moriarty, J. E., **City I: Player's Manual,** *NBSIR 73-110*, 149 pages (Mar. 1973). (Available as COM 73-11191 from the National Technical Information Service, Springfield, Va. 22151.)

Morrissey, B. W., Powell, C. J., **Interpolation of refractive index data,** *Appl. Opt.* **12,** No. 7, 1588-1591 (July 1973).

Mosburg, E. R., Jr., **A study of the CW 28-μm water-vapor laser,** *IEEE J. Quantum Electron.* **QE-9,** No. 8, 843-851 (Aug. 1973).

Mullen, L. O., **High and ultra-high vacuum by pumping with cryocooled surfaces,** (Proc. American Institute of Chemical Engineers, Symp. Series on Cryogenic Pumping, Denver, Colo., 1971). Paper in *Vacuum Technology at Low Temperatures,* **125,** No. 68, 24-30 (1972).

Myers, V. W., **Klein-Gordon equation for a charged particle interacting with an electromagnetic wave,** *J. Franklin Inst. Brief Commun.* **295,** No. 6, 497-499 (June 1973).

Nahman, N. S., **A note on the transition (rise) time versus line length in coaxial cables,** *IEEE Trans. Circuit Theory* **CT-20,** No. 2, 165-167 (Mar. 1973).

Negas, T., Roth, R. S., Parker, H. S., Brower, W. S., **Crystal chemistry of lithium in octahedrally coordinated structures. I. Synthesis of Ba_8 $(Me_6Li_2)O_{24}$(Me = Nb or Ta) and $Ba_{10}(W_6Li_4)$ O_{30}. II. The tetragonal bronze phase in the system BaO-Nb_2O_5-Li_2O,** *J. Solid State Chem.* **8,** No. 1, 1-13 (1973).

Negas, T., **The $SrMnO_{3-x}$-Mn_3O_4 system,** *J. Solid State Chem.* **7,** 85-88 (1973).

Nelson, J. D., Blair, W., Brinckman, F. E., Colwell, R. R., Iverson, W. P., **Biodegradation of phenylmercuric acetate by mercury-resistant bacteria,** *Appl. Microbiol.* **26,** No. 3, 321-326 (Sept. 1973).

Newbury, D. E., Yakowitz, H., Myklebust, R. L., **Monte Carlo calculations of magnetic contrast from cubic materials in the scanning electron microscope,** *Appl. Phys. Lett.* **23,** No. 8, 488-490 (Oct. 15, 1973).

Newell, A. C., Baird, R. C., Wacker, P. F., **Accurate measurement of antenna gain and polarization at reduced distances by an extrapolation technique,** *IEEE Trans. Antennas Propagat.* **AP-21,** No. 4, 418-431 (July 1973).

O'Connell, J. S., **Electromagnetic sum rules,** (Proc. Int. Conf. on Photonuclear Reactions and Applications, Pacific Grove, Calif., Mar. 26-30, 1973). Paper in *International Conference on Photonuclear Reactions and Applications*, B. L. Berman, Ed., pp. 71-94 (Ernest O. Lawrence Livermore Laboratory, University of California, Livermore, Calif., 1973). (Available as CONF-730301 from the National Technical Information Service, Springfield, Va. 22151).

O'Connell, J. S., **Neutrino disintegration of the deuteron,** (Proc. Int. Conf. on Few Particle Problems in the Nuclear Interaction, Los Angeles, Calif., Aug. 28-Sept. 1, 1972). Paper in *Few Particle Problems in the Nuclear Interaction*, I. Slaus, S. A. Moszkowski, R. P. Haddock, and W. T. H. van Oers, Eds., pp. 906-909 (North Holland Publishing Co., Amsterdam, The Netherlands, 1972).

Ogburn, F., Johnson, C. E., **Banded structure of electroless nickel,** *Plating Technical Brief*, pp. 1043-1044 (Oct. 1973).

Okabe, H., Dibeler, V. H., **Photon impact studies of C_2HCN and CH_3CN in the vacuum ultraviolet; heats of formation of C_2H and CH_3CN,** *J. Chem. Phys.* **59,** No. 5, 2430-2435 (Sept. 1, 1973).

Okabe, H., Splitstone, P. L., Ball, J. J., **Ambient and source SO_2 detector based on a fluorescence method,** *J. Air Pollut. Contr. Ass.* **23,** No. 6, 514-516 (June 1973).

Olf, H. G., Fanconi, B., **Low frequency Raman-active lattice vibrations of *n*-paraffins,** *J. Chem. Phys.* **59,** No. 1, 534-544 (July 1, 1973).

Olien, N. A., Sarkes, L. A., **Keeping up with LNG—a new literary awareness service helps to do the job,** *Amer. Gas Ass. Mon.* **55,** No. 7-8, 29 and 31 (July-Aug. 1973).

Olson, W. B., **A precision photoelectric azimuthal polarimeter,** *Opt. Eng.* **12,** No. 3, 102-105 (May/June 1973).

Omont, A., Smith, E. W., Cooper, J., **Redistribution of resonance radiation. I. The effect of collisions,** *Astrophys. J.* **175,** No. 1, Part 1, 185-199 (July 1972).

Omont, A., Smith, E. W., Cooper, J., **Redistribution of resonance radiation. II. The effect of magnetic fields,** *Astrophys. J.* **182,** No. 1, Part 1, 283-300 (May 15, 1973).

Orcutt, R. H., **Interlot density variation of a siloxane manometer fluid,** *J. Vac. Sci. Technol.* **10,** No. 4, p. 506 (July-Aug. 1973).

Ordman, E. T., **Convergence and abstract spaces in functional analysis,** *J. Undergraduate Math.* **1,** No. 2, 79-96 (Sept. 1969); **2,** No. 1, 25-36 (Mar. 1970).

Ott, W. R., Fieffe-Prevost, P., Wiese, W. L., **VUV radiometry with hydrogen arcs. 1: Principle of the method and comparisons with blackbody calibrations from 1650 Å to 3600 Å,** *Appl. Opt.* **12,** No. 7, 1618-1629 (July 1973).

Ott, W. R., Wiese, W. L., **Far ultraviolet spectral radiance calibrations at NBS,** *Opt. Eng.* **12,** No. 3, 86-94 (May/June 1973).

Paabo, M., **Analytical methods for the detection of toxic elements in dry paint matrices—a literature survey,** *NBSIR 73-251*, 49 pages (July 1973). (Available as PB 224688 from the National Technical Information Service, Springfield, Va. 22151.)

Page, C. H., **Logarithmic quantities and units,** *Proc. IEEE Letters* **61,** No. 10, 1516-1517 (Oct. 1973).

Parker, R. L., **Crystal growth,** Article in *Yearbook of Science and Technology, McGraw-Hill Encyclopedia of Science and Technology*, pp. 158-160 (McGraw-Hill Book Co., New York, N.Y., 1973).

Peiser, H. S., Corruccini, R. J., Newman, S. B., **Standardization and measurement services in Turkey,** *NBSIR 73-172*, 90 pages (Oct. 14-28, 1972). (Available as COM 73-11175 from the National Technical Information Service, Springfield, Va. 22151.)

Peiser, H. S., Coyle, T. D., Eby, R. K., **A report on a survey in Korea on standardization and measurement services in support of industrialization goals of Korea.** *NBSIR 73-185*, 41 pages, (June 19-30, 1972).

Pella, P. A., DeVoe, J. R., **Internal standardization in Mössbauer spectrometry,** *Appl. Spectrosc.* **25,** No. 4, 472-474 (July/Aug. 1971).

Penn, D. R., **The concept of the surface molecule in chemisorption,** *Surface Sci.* **39,** 333-340 (1973).

Petersen, F. R., McDonald, D. G., Cupp, J. D., Danielson, B. L., **Rotational constants for $^{12}C^{16}O_2$ from beats between Lamb-dip-stabilized lasers,** *Phys. Rev. Lett.* **31,** No. 9, 573-576 (Aug. 27, 1973).

Peterson, R. L., **Kinks in the magnetophonon oscillations of semiconductors,** *Phys. Rev. B* **7,** No. 12, 5405-5408 (June 15, 1973).

Pezoldt, V. J., Persensky, J. J., **Human factors evaluation of a voice encoding system,** *NBSIR 73-131,* 57 pages (Mar. 1973). (Available as COM 73-10863 from the National Technical Information Service, Springfield, Va. 22151.)

Phelan, R. J., Jr., Cook, A. R., **Electrically calibrated pyroelectric optical-radiation detector,** *Appl. Opt.* **12,** No. 10, 2494-2500 (Oct. 1973).

Phelps, A. V., **Collision cross sections for electrons with atmospheric species,** (Proc. IAGA Symp. on Aurora and Airglow, Moscow, Aug. 1971), *Ann. Geophys.* **28,** No. 3, 611-625 (1972).

Pierce, E. T., Kelly, K. L., McPherson, M. A., Howett, G. L., Booker, R. L., **Emergency vehicle warning devices: Interim review of the state-of-the-art relative to performance standards,** *LESP-RPT-0501-00,* 73 pages. Available as PB211938 from the National Technical Information Service, Springfield, Va. 22151, (U.S. Department of Justice, Law Enforcement Assistance Administration, National Institute of Law Enforcement and Criminal Justice, Washington, D.C., May 1972).

Piganiol, M. P., Balke, S., Branscomb, L. M., Hamada, S., Hookway, H. T., Tell, B. V., **Ad Hoc Group on Scientific and Technical Information OECD, INFORMATION for a changing society, some policy considerations,** *Organisation for Economic Co-operation and Development,* 46 pages (OECD Publications, Paris, France, 1971).

Pike, R. G., Hay, R. E., Clifton, J. R., Beeghly, H. F., Mathey, R. G., **Nonmetallic coatings for concrete reinforcing bars,** *Public Roads* **37,** No. 5, 185-197 (June 1973).

Powell, C. J., **Contrasting valence-band Auger-electron spectra for silver and aluminum,** *Phys. Rev. Lett.* **30,** No. 23, 1179-1182 (June 4, 1973).

Powell, C. J., **High-resolution measurements of Auger-electron and photoelectron structure in the secondary-electron energy distributions of aluminum, nickel and copper,** (Proc. Int. Conf. on Inner Shell Ionization Phenomena and Future Applications, Atlanta, Ga., Apr. 17-22, 1972), Paper in *Proceedings of the International Conference on Inner Shell Ionization Phenomena and Future Applications,* R. W. Fink, S. T. Manson, J. M. Palms and P. V. Rao, Eds., CONF-720404, 1, 743-756 (U.S. Atomic Energy Commission, Technical Information Center, Oak Ridge, Tenn., Jan. 1973).

Powell, C. J., **Semiautomated data-recording and control system for an electron energy analyzer,** *Rev. Sci. Instrum.* **44,** No. 8, 1031-1033 (Aug. 1973).

Powell, R. L., Hall, W. J., Hust, J. G., **The fitting of resistance thermometer data by orthogonal functions,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry,* H. H. Plumb, Editor-in-Chief, **4,** Part 2, 1423-1431 (Instrument Society of America, Pittsburgh, Pa., 1972).

Powell, R. L., **Revision of the standard reference data for thermocouples,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry,* H. H. Plumb, Editor-in-Chief, **4,** Part 3, 1579-1584 (Instrument Society of America, Pittsburgh, Pa., 1972).

Prince, E., Donnay, G., Martin, R. F., **Neutron diffraction refinement of an ordered orthoclase structure,** *Amer. Mineral.* **58,** 500-507 (1973).

Quindry, T. L., Flynn, D. R., **On a simplified field measurement of noise reduction between spaces,** (Proc. 1973 International Noise Control Engineering Conference, Copenhagen, Denmark, Aug. 22-24, 1973), Paper in *Inter-Noise 73 Proceedings,* O. J. Pedersen, Ed., pp. 199-207 (Inter-Noise 73, Technical University, Lyngby, Denmark, 1973).

Quigley, D. F., **USAC transferability,** *NBSIR 73-183,* 71 pages (Apr. 1973). (Available as COM 73-11177 from the National Technical Information Service, Springfield, Va. 22151.)

Quindry, T. L., **Acoustical evaluation of a single family attached steel-frame modular housing system constructed on an Operation BREAKTHROUGH prototype site,** *NBSIR 73-190,* 21 pages (Apr. 1973). (Available as PB 221189 from the National Technical Information Service, Springfield, Va. 22151.)

Quindry, T. L., **Acoustical evaluation of a single family attached wood-frame modular housing system constructed on an Operation BREAKTHROUGH prototype site,** *NBSIR 73-191,* 21 pages, (Apr. 1973). (Available as PB 221695 from the National Technical Information Service, Springfield, Va. 22151.)

Quindry, T. L., **Acoustical evaluation of a single family detached honeycomb panel housing system constructed on an Operation BREAKTHROUGH prototype site,** *NBSIR 73-192,* 19 pages (Apr. 1973). (Available as PB 222231 from the National Technical Information Service, Springfield, Va. 22151.)

Quinn, T. J., Lee, R. D., **Vacuum tungsten strip lamps with improved stability as radiance temperature standards,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry,* H. H. Plumb, Editor-in-Chief, **4,** Part 1, 395-411 (Instrument Society of America, Pittsburgh, Pa., 1972).

Reader, J., **Comment on: How to mount a Pellin-Broca prism for laser work,** *Appl. Opt.* **12,** No. 7, 1405 (July 1973).

Reader, J., Epstein, G., **Zeeman effect and revised analysis of singly ionized rubidium (Rb II),** *J. Opt. Soc. Amer.* **63, No. 9, 1153-1167 (Sept. 1973).**

Reed, R. P., Arvidson, J. M., Durcholz, R. L., **Tensile properties of polyurethane and polystyrene foams from 76 to 300 K,** (Proc. 1972 Cryogenic Engineering Conf., Boulder, Colo., Aug. 9-11, 1972), Paper E-3 in *Advances in Cryogenic Engineering* **18,** 184-193 (Plenum Press, New York, N.Y., 1973).

Reed, R. P., Schramm, R. E., Clark, A. F., **Mechanical, thermal, and electrical properties of selected polymers,** *Cryogenics* **13,** No. 2, 67-82 (Feb. 1973).

Reichard, T. W., Greene, W. E., Jr., Cattaneo, L. E., Masters, L. W., **Structural tests on housing components of glass fiber reinforced polyester laminate,** *NBSIR 73-188,* 98 pages (Apr. 1973). (Available as PB 221183 from the National Technical Information Service, Springfield, Va. 22151.)

Reno, R. C., Swartzendruber, L. J., **Origin of Mössbauer linewidth in stainless steel,** (Proc. 18th AIP Conf. on Magnetism and Magnetic Materials, Denver, Colo., Nov. 28-Dec. 1, 1972), Paper in *Magnetism and Magnetic Materials*, C. D. Graham, Jr., and J. J. Rhyne, Eds., No. 10, 1350-1353 (American Institute of Physics, New York, N.Y., 1973).

Richmond, J. C., Walters, E. L., **Development of a standard for passive night vision devices, a progress report,** (Proc. Carnahan Conf. on Electronic Crime Countermeasures, Lexington, Ky., Apr. 25-27, 1973), Paper in *Proceedings 1973 Carnahan Conference on Electronic Crime Countermeasures*, UKY Bull. 102, 145-156 (Apr. 1973).

Roberts, J. R., Andersen, T., Sørensen, G., **Determination of atomic lifetimes and absolute oscillator strengths for neutral and ionized titanium,** *Astrophys. J.* **181,** 567-586 (Apr. 15, 1973).

Roberts, J. R., Andersen, T., Sørensen, G., **Determination of atomic lifetimes and absolute oscillator strengths for neutral and ionized titanium,** *Nucl. Instrum. Methods* **110,** 119-125 (1973).

Roberts, J. R., Andersen, T., Sørensen, G., **Determination of atomic lifetimes and absolute oscillator strengths for neutral and ionized vanadium,** *Astrophys. J.* **181,** 587-604 (Apr. 15, 1973).

Robertson, A. F., Rappaport, M. W., **Fire extinguishment in oxygen enriched atmospheres,** *NASA CR-121150*, 60 pages (National Aeronautics and Space Administration, Washington, D.C., Feb. 1973).

Robertson, A. F., **Tests indicate venting increases smoke from some polymerics,** *Fire Eng.* **126,** No. 9, 97-98 (Sept. 1973).

Robertson, B., **Introduction to field operators in quantum mechanics,** *Amer. J. Phys.* **41,** 678-690 (May 1973).

Roder, H. M., **A new phase transition in solid hydrogen?,** *Cryogenics Letter to Editors* **13,** No. 7, 439-440 (July 1973).

Rosenstock, H. M., Larkins, J. T., Walker, J. A., **Interpretation of photoionization thresholds: Quasiequilibrium theory and the fragmentation of benzene,** *Int. J. Mass Spectrom. Ion Phys.* **11,** 309-328 (1973).

Rossiter, W. J., Jr., **Outdoor performance of plastics. Z. Final update of weathering data,** *NBSIR 73-146*, 103 pages (Mar. 1973). (Available as COM 73-10989 from the National Technical Information Service, Springfield, Va. 22151.)

Roszman, L. J., Hooper, C. F., Jr., **Distribution of the time-dependent microfield in a plasma,** *Phys. Rev. A* **7,** No. 6, 2121-2130 (June 1973).

Roth, R. S., Parker, H. S., Brower, W. S., Waring, J. L., **Phase equilibria, crystal chemistry and crystal growth of alkali oxide-metal oxide systems,** (Proc. NATO sponsored Advanced Study Institute, Belgirate, Italy, Sept. 1972), Paper in *Fast Ion Transport in Solids*, pp. 217-232 (North-Holland Publ. Co., Amsterdam, The Netherlands, 1973).

Rowe, J. M., Livingston, R. C., Rush, J. J., **Neutron quasielastic scattering study of SH^- reorientation in the cubic phases of cesium and rubidium hydrosulfide,** *J. Chem. Phys.* **58,** No. 12, 5469-5473 (June 15, 1973).

Rubin, A. I., **The social impact of noise: A survey of medical, psychological, and social consequences,** *Environmental Protection Agency Report No. NTID300.11*, 25 pages (Available as PB 206724 from the National Technical Information Service, Springfield, Va. 22151, Dec. 1971).

Ruff, A. W., Ives, L. K., **On the temperature dependence of stacking fault energy in cubic and hexagonal silver-tin alloys,** *Phys. Status Solidi A*, No. 16, 133-149 (1973).

Rupp, N. W., **Intermediary base and cementation,** Chapter 5 in *Operative Dentistry*, L. Baum, Ed., pp. 59-72 (W. B. Saunders Co., Philadelphia, Pa., 1973).

Rush, J. J., de Graaf, L. A., Livingston, R. C., **Neutron scattering investigation of the rotational dynamics and phase transitions in sodium and cesium hydrosulfides,** *J. Chem. Phys.* **58,** No. 8, 3439-3448 (Apr. 15, 1973).

Saks, T. H., Yates, R. F., Goodman, K. M., **The Shirley Highway Express-Bus-on-Freeway demonstration project-users' reactions to innovative bus features,** *NBSIR 73-265*, 53 pages (June 1973). (Available as COM 73-11453 from the National Technical Information Service, Springfield, Va. 22151).

Sanchez, I. C., Colson, J. P., Eby, R. K., **Theory and observations of polymer crystal thickening,** *J. Appl. Phys.* **44,** No. 10, 4332-4339 (Oct. 1973).

Santoro, A., Mighell, A. D., **Coincidence-site lattices,** *Acta Crystallogr.* **A29,** Part 2, 169-175 (Mar. 1973).

Santoro, A., Mighell, A. D., **Properties of crystal lattices: The derivative lattices and their determination,** *Acta Crystallogr.* **A28,** Part 3, 284-287 (May 1972).

Schneider, S. J., Levin, E. M., **Polymorphism of K_2CO_3,** *J. Amer. Ceram. Soc. – Discussions and Notes* **56,** No. 4, 218-219 (1973).

Schooley, J. F., **Enhancement effects,** (Proc. Summer Course on The Science and Technology of Superconductivity, held at Georgetown University, Washington, D.C., Aug. 13-26, 1971), Paper in *The Science and Technology of Superconductivity*, W. D. Gregory, W. N. Mathews, Jr., and E. A. Edelsack, Eds., **1,** 405-428 (Plenum Press, New York, N.Y., 1973).

Schooley, J. F., Soulen, R. J., Jr., **The use of superconductors to provide fixed points on a cryogenic temperature scale,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4,** Part 1, 169-174 (Instrument Society of America, Pittsburgh, Pa., 1972).

Schooley, J. F., **Superconductors in thermometry,** (Proc. Summer Course on The Science and Technology of Superconductivity held at Georgetown University, Washington, D.C., Aug. 13-26, 1971), Paper in *Science and Technology of Superconductivity*, W. D. Gregory, W. N. Mathews, Jr., and E. A. Edelsack, Eds., **2,** 625-630 (Plenum Press, New York, N.Y., 1973).

Schroder, I. G., **Emission of long-range α-particles in the subthermal-, thermal; and resonance-neutron fission of ^{239}Pu,** Nucl. Phys. **A195,** 257-268 (1972).

Seltzer, S. M., Berger, M. J., Rosenberg, T. J., **Auroral bremsstrahlung at balloon altitudes,** *NASA Spec. Publ. 3081*, 25 pages (National Aeronautics and Space Administration, Washington, D.C., 1973).

Shaver, J. R., Masters, L. W., Reichard, T. W., Pielert, J. H., **Environmental evaluation of polyurethane foam core sandwich panel construction,** *NBSIR 73-105*, 47 pages (Dec.

1972). (Available as PB 219118 from the National Technical Information Service, Springfield, Va. 22151.)

Shirley, J. H., **Semiclassical theory of saturated absorption in gases,** *Phys. Rev. A* **8,** No. 1, 347-368 (July 1973).

Shoub, H., Son, B. C., **Fire endurance tests of plywood on steel joist floor assemblies, with and without ceiling,** *NBSIR 73-141*, 37 pages (Mar. 1972). (Available as PB 220430/3 from the National Technical Information Service, Springfield, Va. 22151.)

Simmons, J. D., Keller, R. A., **Interferometric effects on the output of organic dye lasers,** *Appl. Opt.* **12,** No. 9, 2033 (Sept. 1973).

Sindt, C. F., **Insulation of liquid oxygen dewars,** *NBSIR 73-308*, 44 pages (Apr. 1973). (Available as AD 763325 from the National Technical Information Service, Springfield, Va. 22151.)

Siu, C. I., Ellis, W. M., Kusuda, T., **Test of a polyester composite wall panel for moisture accumulation and potential removal of moisture through venting,** *NBSIR 73-220*, 18 pages (May 1973). (Available as PB 222437 from the National Technical Information Service, Springfield, Va. 22151.)

Siu, M. C. I., **Equations for thermal transpiration,** *J. Vac. Sci. Technol.* **10,** No. 2, 368-372 (Mar./Apr. 1973).

Slattery, W. J., **Standards information for your company,** *Stand. Eng.* **25,** No. 3, p. 22, (June 1973).

Smith, M. W., Martin, G. A., Wiese, W. L., **Systematic trends and atomic oscillator strengths,** *Nucl. Instrum. Methods* **110,** 219-226 (1973).

Smith, R. L., Honda, T., **Impulse rise and fall times of biplanar vacuum photodiodes,** *Appl. Opt.* **12,** No. 7, 1606-1610 (July 1973).

Smith, R. L., Phelan, R. J., Jr., **Limitations of the use of vacuum photodiodes in instruments for the measurement of laser power and energy,** *Appl. Opt.* **12,** No. 4, 795-798 (Apr. 1973).

Somes, N. F., **Abnormal loading on buildings and progressive collapse,** *NBSIR 73-221*, 76 pages (May 1973). (Available as PB 220849 from the National Technical Information Service, Springfield, Va. 22151.)

Son, B. C. **Fire endurance test of a fiber glass reinforced polyester double wall assembly,** *NBSIR 73-168*, 27 pages (Apr. 1973). (Available as PB 221184 from the National Technical Information Service, Springfield, Va. 22151.)

Son, B. C., **Fire endurance test of an interdwelling double wall constructed of polyurethane foam-filled sandwich panels,** *NBSIR 73-170*, 27 pages (Apr. 1973). (Available as PB 221193 from the National Technical Information Service, Springfield, Va. 22151.)

Son, B. C., **Fire endurance test of an interdwelling double wall construction of paper honeycomb and gypsum board,** *NBSIR 73-166*, 24 pages (Apr. 1973). (Available as PB 222297 from the National Technical Information Service, Springfield, Va. 22151.)

Son, B. C., **Fire endurance test of a roof/ceiling construction of paper honeycomb and gypsum board,** *NBSIR 73-167*, 20 pages (Jan. 1973). (Available as PB 222298 from the National Technical Information Service, Springfield, Va. 22151.)

Son, B. C., **Fire endurance test of a steel sandwich panel floor construction,** *NBSIR 73-164*, 25 pages (Apr. 1973). (Available as PB 221642 from the National Technical Information Service, Springfield, Va. 22151.)

Son, B. C., **Fire endurance test of a wood stud interdwelling double wall construction,** *NBSIR 73-169*, 21 pages (Apr. 1973). (Available as PB 221185 from the National Technical Information Service, Springfield, Va. 22151.)

Son, B. C., **Fire endurance test of plywood-faced exterior walls for single family housing,** *NBSIR 73-140*, 24 pages (Mar. 1973). (Available as PB 220226/5 from the National Technical Information Service, Springfield, Va. 22151.)

Son, B. C., **Fire endurance test on a steel tubular column protected with gypsum board,** *NBSIR 73-165*, 23 pages (Apr. 1973). (Available as PB 221474 from the National Technical Information Service, Springfield, Va. 22151.)

Son, B. C., **Fire endurance tests of steel sandwich panel exterior wall and roof/ceiling constructions,** *NBSIR 73-135*, 38 pages (Dec. 1972). (Available as PB 221310 from the National Technical Information Service, Springfield, Va. 22151.)

Son, B. C., Shoub, H., **Fire endurance tests of double module walls of gypsum board and steel studs,** *NBSIR 73-173*, 32 pages (Apr. 1973). (Available as COM 73-10844 from the National Technical Information Service, Springfield, Va. 22151.)

Soulen, R. J., Jr., Schooley, J. F., **Superconducting transition widths and reproducibilities,** (Proc. XIIIth Int. Congress of Refrigeration, Washington, D.C., Aug. 27-Sept. 3, 1971), Paper in *Progress in Refrigeration Science and Technology* **1,** 399-403 (Avi Publ. Co., Inc., Westport, Conn., 1973).

Sparks, L. L., Powell, R. L., **Calibration of capsule platinum resistance thermometers at the triple point of water,** (Proc. 5th Symp. on Temperature, Its Measurement and Control in Science and Industry, Washington, D.C., June 21-24, 1971), Paper in *Temperature, Its Measurement and Control in Science and Industry*, H. H. Plumb, Editor-in-Chief, **4,** Part 2, 1415-1421 (Instrument Society of America, Pittsburgh, Pa., 1972).

Spurgeon, J. C., **Response characteristics of a portable x-ray fluorescence lead detector: Detection of lead in paint,** *NBSIR 73-231*, 37 pages (June 1973). (Available as PB 224645 from the National Technical Information Service, Springfield, Va. 22151.)

Stabler, T. M., **U.S. membership in the International Organization of Legal Metrology,** *NBSIR 73-159*, 12 pages (Apr. 1973). (Available as COM 73-11174 from the National Technical Information Service, Springfield, Va. 22151.)

Stein, P. G., Boyle, D. R., **How to cope with nickel and diming in your minicomputer,** (Proc. IEEE Minicomputer Conf. on Trends and Applications, National Bureau of Standards, Gaithersburg, Md., Apr. 4, 1973), *Symposium Record*, IEEE Catalog No. 73CHO 760-9 C, pp. 5-7 (1973).

Stephenson, J. C., **Vibrational energy transfer in NO,** *J. Chem. Phys.* **59,** No. 3, 1523-1527 (Aug. 1, 1973).

Stephenson, J. C., **Vibrational excitation and relaxation of the CO($v=1$) and CO($v=2$) states,** *Appl. Phys. Lett.* **22,** No. 11, 576-578 (June 1, 1973).

Stillman, R. B., **The concept of weak substitution in theorem-proving,** *J. Ass. Comput. Mach.* **20,** No. 4, 648-667 (Oct. 1973).

Stoltenberg, R. E., **RF null detector NBS/SND,** *NBSIR 73-302*, 88 pages (June 1973). (Available as COM 73-10869 from the National Technical Information Service, Springfield, Va. 22151.)

Straty, G. C., Younglove, B. A., **Some sound velocity measurements on liquid fluorine,** *J. Chem. Phys.* **58,** No. 5, 2191-2192 (Mar. 1, 1973).

Straty, G. C., Younglove, B. A., **Velocity of sound in saturated and compressed fluid oxygen,** *J. Chem. Thermodyn.* **5,** No. 3, 305-312 (May 1973).

Sugar, J., **Ionization energies of the neutral actinides,** *J. Chem. Phys.* **59,** No. 2, 788-791 (July 15, 1973).

Sugar, J., Reader, J., **Ionization energies of doubly and triply ionized rare earths,** *J. Chem. Phys.* **59,** No. 4, 2083-2089 (Aug. 15, 1973).

Sullivan, D. B., **Precise electrical measurements at low temperature,** *Proc. 1972 Applied Superconductivity Conf., Annapolis, Md., May 1-3, 1972,* pp. 631-639 (IEEE, Inc., New York, N.Y., 1972).

Swartzendruber, L. J., Evans, B. J., **Local interactions and spin transfer mechanisms in the Heusler-type alloys $Pd_{1+x}MnSb$ and $Pd_2MnSb_{0.9}Sn_{0.1}$,** (Proc. 18th AIP Conf. on Magnetism and Magnetic Materials, Denver, Colo., Nov. 28-Dec. 1, 1972), Paper in *Magnetism and Magnetic Materials,* C. D. Graham, Jr., and J. J. Rhyne, Eds., No. 10, 1369-1373 (American Institute of Physics, New York, N.Y., 1973).

Tate, E. L., **Anglo-American code implementation,** Paper in *Advances in Librarianship* **3,** 167-194 (Seminar Press, Inc., New York and London, 1972).

Thurber, W. R., Lewis, D. C., Bullis, W. M., **Resistivity and carrier lifetime in gold-doped silicon,** *NBSIR 73-128,* 56 pages (Jan. 31, 1973). (Available as AD 760150 from the National Technical Information Service, Springfield, Va. 22151.)

Tilford, C. R., **A fringe counting laser interferometer manometer,** *Rev. Sci. Instrum.* **44,** No. 2, 180-182 (Feb. 1973).

Tsai, D. H., MacDonald, R. A., **Second sound in a solid under shock compression,** *J. Phys. C: Solid State Phys. Letter to Editor* **6,** L171-L175 (1973).

Verdier, P. H., **Fluctuations in autocorrelation functions in diffusing systems,** Chapter in *Stochastic Processes in Chemical Physics,* K. E. Shuler, Ed., **15,** 137-148 (John Wiley & Sons, Inc., New York, N.Y. 1969).

Waksman, D., Skoda, L. F., Clark, E. J., Godette, M., **Evaluation of lead paint hazard elimination methods, Part II,** *NBSIR 73-127,* 77 pages (Mar. 1973).

Waterstrat, R. M., Dickens, B., **The crystal structure of V_3Rh_5,** *J. Less-Common Metals* **31,** 61-67 (1973).

Waterstrat, R. M., Manuszewski, R. C., **The chromium-iridium constitution diagram,** *J. Less-Common Metals* **32,** 79-89 (1973).

Waterstrat, R. M., Manuszewski, R. C., **The chromium-rhodium constitution diagram,** *J. Less-Common Metals* **32,** 331-343 (1973).

Waterstrat, R. M., **The chromium-platinum constitution diagram,** *Met. Trans.* **4,** 1585-1592 (June 1973).

Waxman, M., Hilsenrath, J., Chen, W. T., **Compressibility factor and the second virial coefficient of boron trifluoride from 0 to 225 °C and for pressures up to 250 bar,** *J. Chem. Phys.* **58,** No. 9, 3692-3701 (May 1, 1973).

Weisman, I. D., **^{109}Ag Knight shift in an $Ag_{0.96}Al_{0.04}$ alloy,** *J. Chem. Eng. Data* **18,** No. 2, 146 (Apr. 1973).

Wells, J. S., **A stabilized HCN laser for infrared frequency synthesis,** *IEEE Trans. Instrum. Meas. IM-22,* No. 2, 113-118 (June 1973).

Wiederhorn, S. M., **Environmental stress corrosion cracking of glass,** (Proc. Int. Conf. on Corrosion Fatigue, Chemistry, Mechanics and Microstructure, University of Connecticut, Storrs, Conn., June 14-18, 1971), Paper in *Corrosion Fatigue,* **NACE-2,** 731-742 (National Association of Corrosion Engineers, Houston, Texas, 1972).

Wong, C. Y., Tamura, T., Marshak, H., Langsford, A., **Effects of nuclear collectivity on the total neutron cross section,** *Particles Nucl.* **4,** 163-174 (1972).

Yakowitz, H., Jacobs, M. H., Hunneyball, P. D., **Analysis of urban particulates by means of combined electron microscopy and x-ray microanalysis,** *Micron* **3,** 498-505 (1972).

Yancey, C. W., Somes, N. F., **Structural tests of a wood-frame housing module,** *NBSIR 73-121,* 111 pages (Mar. 26, 1973). (Available as COM 73-10860 from the National Technical Information Service, Springfield, Va. 22151.)

Yates, J. T., Jr., Madey, T. E., Dresser, M. J., **Adsorption and decomposition of formaldehyde on tungsten (100) and (111) crystal planes,** *J. Catal.* **30,** No. 2, 260-275 (Aug. 1973).

Yokel, F. Y., Reichard, T. W., **Evaluation of the column connections used in a precast concrete modular housing system,** *NBSIR 73-148,* 90 pages (Mar. 26, 1973). (Available as PB 220366/7 from the National Technical Information Service, Springfield, Va. 22151.)

Yokel, F. Y., **Study of the local resistance of conventional plywood subflooring to concentrated load,** *NBSIR 73-116,* 64 pages (Mar. 26, 1973). (Available as PB 220432 from the National Technical Information Service, Springfield, Va. 22151.)

Zimmerman, J. E., Siegwarth, J. D., **Portable helium dewars for use with super-conducting magnetometers,** *Cryogenics* **13,** No. 3, 158-159 (Mar. 1973).

**Publications with prices and SD Catalog numbers may be purchased directly from the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20202 (foreign: one-fourth additional). Microfiche copies are available from the National Technical Information Service (NTIS), Springfield, Va. 22151. Reprints from outside journals and the NBS Journal of Research may often be obtained directly from the authors.*

U. S. GOVERNMENT PRINTING OFFICE : 1974 O - 526- 390

UNITED STATES
GOVERNMENT PRINTING OFFICE
PUBLIC DOCUMENTS DEPARTMENT
WASHINGTON, D.C. 20402

OFFICIAL BUSINESS
PENALTY FOR PRIVATE USE, $300

POSTAGE AND FEES PAID
U.S. GOVERNMENT PRINTING OFFICE
375

JOURNAL OF RESEARCH of the National Bureau of Standards Vol. 78B, No. 1 Jan.–Mar. 1974
B. Mathematical Sciences

SJRB UNIVERSITY
UNIVERSITY MICROFILMS
JOY REED
300 N ZEEB RD
ANN ARBOR MI 48106

Contents

Page

A class of positive stable matrices. David Carlson. 1

The Smith normal form of a partitioned matrix. Morris Newman. 3

A Lyapunov theorem for angular cones. Charles R. Johnson. 7

Second, third, and fourth order *D*-stability. Charles R. Johnson. 11

Comparison of some FORTRAN programs for matrix inversion. Kenneth E. Fitzgerald. 15

On characters of subgroups. Russell Merris. 35

Integer arithmetic determination of polynomial real roots. George W. Reitwiesner. 39

Publications of the National Bureau of Standards. 45

Library of Congress Catalog Card Number: 62–4414

For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402. Order by SD Catalog No. C 13.22/sec.B:78B (1). Price $2.25 (single copy). Subscription price: $9.00 a year; $2.25 additional for foreign mailing.
UNITED STATES GOVERNMENT PRINTING OFFICE, WASHINGTON : 1974